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The G-stable rank for tensors and the cap set problem

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We introduce the G-stable rank of a higher order tensors over perfect fields. The G-stable rank is related to the Hilbert–Mumford criterion for stability in geometric invariant theory. We will relate the G-stable rank to the tensor rank and slice rank. For numerical applications, we express the G-stable rank as a solution to an optimization problem. Over the field $\mathbb{F}_3$ we discuss an application to the cap set problem.

1. Introduction

1A. ***Ranks of tensors.*** We will introduce the G-stable rank for tensors, describe its properties and relate it to other notions for the rank of a tensor, such as the *tensor rank*, *border rank*, *slice rank* and *noncommutative rank*. Suppose that K is a field, $V_1, V_2, \dots, V_d$ are finite dimensional K-vector spaces and $V = V_1 \otimes V_2 \otimes \cdots \otimes V_d$ is the tensor product. All tensor products are assumed to be over the field K unless stated otherwise. The definition of tensor rank goes back to Hitchcock [1927; 1928].

Definition 1.1. The rank $\mathrm{rk}(v)$ of a tensor $v \in V$ is the smallest nonnegative integer r such that we can write $v = \sum_{i=1}^{r} v_{i,1} \otimes v_{i,2} \otimes \cdots \otimes v_{i,d}$ with $v_{i,j} \in V_j$ for all i and j.

There are many applications of the tensor rank and the related concept of CP-decomposition; see [Kolda and Bader 2009] for a survey. For $d = 2$, tensor rank coincides with matrix rank. Computing the tensor rank is NP-hard [Håstad 1989], and tensor rank is ill-behaved. For example, the set $X(\mathrm{rk}, r) \subseteq V$ of all tensors of rank $\leq r$ is not always Zariski closed. The *border rank* $\mathrm{brk}(v)$ of a tensor v was introduced by Bini [1980] and is the smallest positive integer r such that v lies in the Zariski closure of $X(\mathrm{rk}, r)$; see also [Bürgisser et al. 1997; Landsberg 2012]. The *slice rank* of a tensor was introduced by Terence Tao; see [Tao and Sawin 2016; Blasiak et al. 2017].

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Definition 1.2. A nonzero tensor $v \in V$ has slice rank 1 if it is contained in

$$V_1 \otimes \cdots \otimes V_{i-1} \otimes w \otimes V_{i+1} \otimes \cdots \otimes V_d$$

for some i and some $w \in V_i$. The slice rank $\mathrm{srk}(v)$ of an arbitrary tensor $v \in V$ is the smallest nonnegative integer r such that v is the sum of r tensors with slice rank 1.

1B. ***The definition of the G-stable rank.*** We will now define the G-stable rank. It was noted in [Blasiak et al. 2017] that the slice-rank is closely related the notion of stability in geometric invariant theory; see [Mumford et al. 1994]. The authors also introduce the *instability* of a tensor and relate it to the slice rank. The instability of a tensor does not behave like a rank function, but it is closely related to the G-stable rank. We will define the G-stable rank in terms of degenerations and power series. It can also be defined in terms 1-parameter subgroup using the Hilbert–Mumford criterion in geometric invariant theory (see Theorem 2.4). The Hilbert–Mumford criterion is often formulated when working over an algebraically closed field K. Kempf [1978] showed that the Hilbert–Mumford criterion still applies when working of a perfect field K. For this reason, we will assume that K is a perfect field for the remainder of the paper.

To define the G-stable rank, we need to introduce the ring $K[\![t]\!]$ of formal power series in t and its quotient field $K(\!(t)\!)$ of formal Laurent series. The t-valuation of a series $a(t) \in K(\!(t)\!)$ is the smallest integer d such that $a(t) = t^d b(t)$ with $b(t) \in K[\![t]\!]$. By convention, $\mathrm{val}_t(0) = \infty$. If W is a K-vector space and $v(t) \in K(\!(t)\!) \otimes W$ then we define

$$\mathrm{val}_t(v(t)) = \min\{d \mid v(t) = t^d w(t) \text{ and } w(t) \in K[\![t]\!] \otimes W\}.$$

We say that $v(t)$ has no poles when $\mathrm{val}_t(v(t)) \geq 0$, which is equivalent to $v(t) \in K[\![t]\!] \otimes W$. In that case we say that $\lim_{t\to 0} v(t)$ exists, and is equal to $v(0) \in W$.

The group $\mathrm{GL}(W, K(\!(t)\!))$ will denote the group of $K(\!(t)\!)$-linear endomorphisms of the space $K(\!(t)\!) \otimes_K W$. We may view $\mathrm{GL}(W, K(\!(t)\!))$ as a subset of $K(\!(t)\!) \otimes_K \mathrm{End}(W)$. If $W = K^n$ then $K(t) \otimes_K W \cong K(\!(t)\!)^n$ and we can identify $\mathrm{GL}(W, K(\!(t)\!))$ with the set of $n \times n$ matrices with entries in the field $K(\!(t)\!)$. If $R \subseteq K(\!(t)\!)$ is a K-subalgebra of $K(\!(t)\!)$ (such as $R = K[\![t]\!]$, $R = K[t, t^{-1}]$ or $R = K[t]$), then $\mathrm{GL}(W, R)$ is the intersection of $\mathrm{GL}(W, K(\!(t)\!))$ with $R \otimes_K \mathrm{End}(W)$ in $K(\!(t)\!) \otimes_K \mathrm{End}(W)$. Note that the inverse of an element in $\mathrm{GL}(W, R)$ lies in $\mathrm{GL}(W, K(\!(t)\!))$, but not necessarily in $\mathrm{GL}(W, R)$. If $W = K^n$, then $\mathrm{GL}(W, R)$ is the set of $n \times n$ matrices with entries in R that, viewed as a matrix with entries in $K(\!(t)\!)$, are invertible.

We consider the action of the group $G = \mathrm{GL}(V_1) \times \mathrm{GL}(V_2) \times \cdots \times \mathrm{GL}(V_d)$ on the tensor product space $V = V_1 \otimes V_2 \otimes \cdots \otimes V_d$. For any K-subalgebra $R \subseteq K(\!(t)\!)$, we define

$$G(R) = \mathrm{GL}(V_1, R) \times \cdots \times \mathrm{GL}(V_d, R).$$

The group $G(K(\!(t)\!))$ acts on $K(\!(t)\!) \otimes V$.

For any weight $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{R}^d_{>0}$ we will have a notion of G-stable rank, but the case $\alpha = (1, 1, \ldots, 1)$ will be of particular interest. Suppose that $g(t) \in G(K[\![t]\!])$, $v \in V$ and $\mathrm{val}_t(g(t) \cdot v) > 0$.

We consider the slope

$$\mu_\alpha(g(t), v) = \frac{\sum_{i=1}^{d} \alpha_i \operatorname{val}_t(\det g_i(t))}{\operatorname{val}_t(g(t)\cdot v)}. \tag{1}$$

Heuristically, the denominator in the slope measures how fast $g(t)\cdot v$ goes to 0 as $t \to 0$. The numerator measures how fast the product of the eigenvalues of $g_1(t), g_2(t), \dots, g_d(t)$ go to 0 as $t \to 0$. A small slope means that v is very unstable in the sense that $g(t)\cdot v$ goes to 0 quickly, while, on average, the eigenvalues of $g_i(t)$ go to 0 slowly.

Definition 1.3. The G-stable α-rank $\operatorname{rk}_\alpha^G(v)$ of v as the infimum of all $\mu_\alpha(g(t), v)$ where $g(t) \in G(K[\![t]\!])$ and $\operatorname{val}_t(g(t)\cdot v) > 0$. If $\alpha = (1, 1, \dots, 1)$, then we may write rk^G instead of $\operatorname{rk}_\alpha^G$.

Using a K-rational version of the Hilbert–Mumford criterion [Hilbert 1893; Mumford et al. 1994] by Kempf [1978], we will show that for computing the G-stable α-rank, one only has to consider $g(t)$ that are 1-parameter subgroups of G without poles (Theorem 2.4). In this context, $g(t) \in G(K[t])$ is a 1-parameter subgroup if for every i we can choose a basis of V_i such that the matrix of $g(t)$ is diagonal and each diagonal entry of that matrix is a nonnegative power of t.

We denote the standard basis vectors in K^n by $[1], [2], \dots, [n]$, and we abbreviate a tensor $[i_1] \otimes [i_2] \otimes \cdots \otimes [i_d]$ by $[i_1, i_2, \dots, i_d]$.

Example 1.4. Suppose that $V_1 = V_2 = V_3 = K^2$, and $v = [2, 1, 1] + [1, 2, 1] + [1, 1, 2]$. We take $g(t) = (g_1(t), g_2(t), g_3(t))$ with

$$g_1(t) = g_2(t) = g_3(t) = \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}.$$

We have $g(t)\cdot v = t^2 v$, $\det(g_i(t)) = t$, and

$$\mu(g(t), v) = \mu_{(1,1,1)}(g(t), v) = \frac{\operatorname{val}_t(\det g_1(t)) + \operatorname{val}_t(\det g_2(t)) + \operatorname{val}_t(\det g_3(t))}{\operatorname{val}_t(g(t)\cdot v)} = \frac{1+1+1}{2} = \frac{3}{2}.$$

This shows that $\operatorname{rk}^G(v) \le \frac{3}{2}$. One can show that $\operatorname{rk}^G(v) = \frac{3}{2}$; see Examples 1.5 and 4.5.

1C. *Properties of the G-stable rank.* If v is a rank 1 tensor, then we have $\operatorname{rk}_\alpha^G(v) = \min\{\alpha_1, \dots, \alpha_d\}$ and $\operatorname{rk}^G(v) = 1$ (Lemma 3.1). The G-stable rank is related to other notions of rank. We have (see Corollary 3.7 and Proposition 4.9)

$$\frac{2\operatorname{srk}(v)}{d} \le \operatorname{rk}^G(v) \le \operatorname{srk}(v) \le \operatorname{brk}(v) \le \operatorname{rk}(v).$$

This implies that for $d = 2$, the G-stable rank, the slice rank and the matrix rank coincide.

The tensor rank depends on the field one is working over. For example, the tensor $[1, 1, 1] - [1, 2, 2] - [2, 1, 2] - [2, 2, 1]$ has rank 3 as a tensor in $\mathbb{R}^{2\times 2\times 2}$ but rank 2 when viewed as a tensor in $\mathbb{C}^{2\times 2\times 2}$. Although it is not clear from the definition, the G-stable rank does not change when passing to a field extension of K (see Theorem 2.5).

Another nice property of the G-stable rank is that the border rank phenomenon does not happen and the set $X(\mathrm{rk}^G_\alpha, r)$ of all tensors v with $\mathrm{rk}^G_\alpha(v) \le r$ is Zariski closed (Theorem 2.11). Tao and Sawin [2016] proved a similar result for the slice rank, and this implies that $\mathrm{srk}(v) \le \mathrm{brk}(v)$ for all tensors v.

Like other rank notions, the G-stable rank satisfies the triangle inequality: $\mathrm{rk}^G_\alpha(v+w) \le \mathrm{rk}^G_\alpha(v) + \mathrm{rk}^G_\alpha(w)$ (see Proposition 3.6). If $v \in V_1 \otimes V_2 \otimes \cdots \otimes V_d$ and $w \in W_1 \otimes W_2 \otimes \cdots \otimes W_d$ then the direct sum of v and w, viewed as

$$\begin{pmatrix} v \\ w \end{pmatrix} \in \begin{array}{c} V_1 \otimes V_2 \otimes \cdots \otimes V_d \\ \oplus \\ W_1 \otimes W_2 \otimes \cdots \otimes W_d \end{array} \subseteq V \boxplus W := \begin{pmatrix} V_1 \\ \oplus \\ W_1 \end{pmatrix} \otimes \begin{pmatrix} V_2 \\ \oplus \\ W_2 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} V_d \\ \oplus \\ W_d \end{pmatrix}$$

will be denoted by $v \boxplus w$. (We will use the notation $v \boxplus w$ and $V \boxplus W$ rather than the more common notation $v \oplus w$ and $V \oplus W$ to emphasize that this direct sum is a "vertical" operation, i.e., the sum $V_i \oplus W_i$ is taken within each tensor factor.) The G-stable rank is additive (Proposition 3.8): $\mathrm{rk}^G_\alpha(v \boxplus w) = \mathrm{rk}^G_\alpha(v) + \mathrm{rk}^G_\alpha(w)$. In particular, if

$$\begin{aligned} v &= [1, 1, \ldots, 1] + [2, 2, \ldots, 2] + \cdots + [r, r, \ldots, r] \\ &= \underbrace{[1, 1, \ldots, 1] \boxplus [1, 1, \ldots, 1] \boxplus \cdots \boxplus [1, 1, \ldots, 1]}_{r} \in \underbrace{K^r \otimes K^r \otimes \cdots \otimes K^r}_{d}, \end{aligned}$$

then $\mathrm{rk}^G_\alpha(v) = r\, \mathrm{rk}^G_\alpha([1, 1, \ldots, 1]) = r \min\{\alpha_1, \ldots, \alpha_d\}$ and $\mathrm{rk}^G(v) = r$. Strassen [1973] conjectured that tensor rank is additive when K is infinite, but Shitov recently gave a counterexample to this long standing conjecture; see [Shitov 2019].

If $v \in V_1 \otimes V_2 \otimes \cdots \otimes V_d$ and $w \in W_1 \otimes W_2 \otimes \cdots \otimes W_e$, then we can form the "horizontal" tensor product $v \otimes w \in V_1 \otimes \cdots \otimes V_d \otimes W_1 \otimes \cdots \otimes W_e$. It is clear that $\mathrm{rk}(v \otimes w) \le \mathrm{rk}(v)\, \mathrm{rk}(w)$. It was recently shown in [Christandl et al. 2019] that we do not always have equality. The G-stable rank behaves quite differently for the horizontal tensor product. We have $\mathrm{rk}^G_{\alpha,\beta}(v \otimes w) = \min\{\mathrm{rk}^G_\alpha(v), \mathrm{rk}^G_\beta(w)\}$ (see Proposition 3.4). If $d = e$ then there is another way of forming a tensor product. The tensor product $v \otimes w$ viewed as

$$\begin{array}{c} v \\ \otimes \\ w \end{array} \in \begin{array}{c} V_1 \otimes V_2 \otimes \cdots \otimes V_d \\ \otimes \\ W_1 \otimes W_2 \otimes \cdots \otimes W_d \end{array} \subseteq \begin{pmatrix} V_1 \\ \otimes \\ W_1 \end{pmatrix} \otimes \begin{pmatrix} V_2 \\ \otimes \\ W_2 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} V_d \\ \otimes \\ W_d \end{pmatrix}$$

will be denoted by $v \boxtimes w$. We will refer to this operation as a *vertical* tensor product or a *Kronecker* tensor product. It is clear that $\mathrm{rk}(v \boxtimes w) \le \mathrm{rk}(v \otimes w)$. It has long been known that $\mathrm{rk}(v \boxtimes w)$ can be smaller than $\mathrm{rk}(v)\, \mathrm{rk}(w)$. For example, if $v_1 = [1, 1, 1] + [2, 2, 1]$, $v_2 = [1, 1, 1] + [2, 1, 2]$ and $v_3 = [1, 1, 1] + [2, 2, 1]$ then $v_1 \boxtimes v_2 \boxtimes v_3$ is the matrix multiplication tensor for 2×2 matrices which has rank 7 [Strassen 1969], so $7 = \mathrm{rk}(v_1 \boxtimes v_2 \boxtimes v_3) < \mathrm{rk}(v_1)\, \mathrm{rk}(v_2)\, \mathrm{rk}(v_3) = 2^3$. If K has characteristic 0, then we have $\mathrm{rk}^G_{\alpha\beta}(v \boxtimes w) \ge \mathrm{rk}^G_\alpha(v)\, \mathrm{rk}^G_\beta(v)$ (Theorem 5.4). We conjecture that this inequality is also true when K is a perfect field of positive characteristic. The slice rank does not behave as nicely with respect to vertical tensor product and $\mathrm{srk}(v \boxtimes w)$ could be larger or smaller than $\mathrm{srk}(v)\, \mathrm{srk}(w)$; see [Christandl et al. 2018, Example 5.2].

1D. ***G-stable rank for complex tensors.*** If $K=\mathbb{C}$, then the G-stable rank can be computed in a different way. For a finite dimensional complex Hilbert space, we will denote the Hermitian form by $\langle\cdot,\cdot\rangle$ and the ℓ_2 norm (or Frobenius norm) by $\|v\|=\sqrt{\langle v,v\rangle}$. Suppose that $V_1,V_2,\dots,V_d$ are finite dimensional Hilbert spaces, which makes V into a Hilbert space. If A is a linear map between finite dimensional Hilbert spaces, then its spectral norm $\|A\|_\sigma$ is the operator norm $\|A\|_\sigma=\max_{v\neq 0}\|Av\|/\|v\|$, which is also the largest singular value of A.

For a tensor $v\in V$, let $\Phi_i(v):(V_1\cdots\otimes\widehat{V_i}\otimes\cdots\otimes V_d)^\star\to V_i$ be the i-th flattening. Then the G-stable α-rank of a tensor $v\in V$ is equal to

$$\mathrm{rk}^G_\alpha(v)=\sup_{g\in G}\min_i\frac{\alpha_i\|g\cdot v\|^2}{\|\Phi_i(g\cdot v)\|_\sigma^2}\tag{2}$$

(see Theorem 5.2).

Example 1.5. Consider again the example $v=[2,1,1]+[1,2,1]+[1,1,2]\in K^{2\times2\times2}$ as in Example 1.4, but now we will work over $K=\mathbb{C}$. We have $\|v\|=\sqrt{3}$. The first flattening of v is equal to

$$\Phi_1(v)=\left(\begin{array}{cc|cc}0&1&1&0\\1&0&0&0\end{array}\right)$$

which has singular values 1 and $\sqrt{2}$. So $\|\Phi_1(v)\|_\sigma=\sqrt{2}$. By symmetry, we also have $\|\Phi_2(v)\|_\sigma=\|\Phi_3(v)\|_\sigma=\sqrt{2}$. It follows that

$$\mathrm{rk}^G(v)=\sup_{g\in G}\min_i\frac{\|g\cdot v\|^2}{\|\Phi_i(g\cdot v)\|_\sigma^2}\geq\min_i\frac{\|v\|^2}{\|\Phi_i(v)\|_\sigma^2}=\frac{3}{2}.$$

1E. ***The cap set problem.*** We say that a subset S of an abelian group A does not contain an arithmetic progression (of length 3) if there are no distinct elements $x,y,z\in S$ with $x+z=2y$. For an abelian group A, let $r_3(A)$ be the largest cardinality of a subset $S\subseteq A$ without an arithmetic progression. Finding upper and lower bounds for $r_3(A)$ has been studied extensively in number theory. For the group $A=(\mathbb{Z}/3\mathbb{Z})^n\cong\mathbb{F}_3^n$ this is known as the cap set problem. Brown and Buhler [1982] showed that $r_3(\mathbb{F}_3^n)=o(3^n)$ and this was later improved to $r_3(\mathbb{F}_3^n)=O(3^n/n)$ by Meshulam [1995] and to $o(3^n/n^{1+\varepsilon})$ by Bateman and Katz [2012]. Using the polynomial method of Croot, Lev and Pach [Croot et al. 2017], who showed that $r_3((\mathbb{Z}/4\mathbb{Z})^n)=o(c^n)$ for some $c<4$, Ellenberg and Gijswijt [2017] showed that $r_3(\mathbb{F}_3^n)\leq 3\theta^n=o(2.756^n)$, where $\theta<2.756$. We also have a lower bound $r_3(\mathbb{F}_3^n)=\omega(2.21^n)$ by Edel. The bound (and the proof) of Ellenberg and Gijswijt is also valid for tricolored sum-free sets for which an asymptotic lower bound $\omega(\theta^n)$ was given by Kleinberg, Sawin and Speyer [Kleinberg et al. 2018]. So for tricolored sum-free sets, the upper and lower bound have the same exponential growth.

Tao noted that the Ellenberg–Gijswijt proof can be nicely presented using the concept of slice rank. A key idea is to prove the inequality $r_3(\mathbb{F}_3^n)\leq\mathrm{srk}(u^{\boxtimes n})$ where

$$u=\sum_{\substack{i,j,k\in\mathbb{Z}/3\mathbb{Z}\\ i+j+k=0}}[i,j,k]\in\mathbb{F}_3^{3\times3\times3}$$

n	upper bound	E.–G.	P.
1	2	3	2
2	6	7	4
3	15	18	11
4	39	45	30
5	105	123	72
6	274	324	196
7	722	822	548
8	1,957	2,277	1,350
9	5,193	6,075	3,686
10	13,770	15,579	10,386

n	upper bound	E.–G.	P.
11	37,477	43,365	25,896
12	100,296	116,532	70,890
13	266,997	300,888	200,592
14	728,661	840,030	503,964
15	1,961,103	2,267,838	1,382,334
16	5,235,597	5,883,309	3,922,206
17	14,316,784	16,459,335	9,906,786
18	38,685,141	44,580,537	27,215,544
19	103,504,935	116,055,423	77,370,282
20	283,466,139	325,182,235	195,202,290

Table 1. Comparison to the bounds of Ellenberg and Gijswijt and of Petrov.

and to combine this with asymptotic estimates for the slice rank. We will show that $r_3(\mathbb{F}_3^n) \leq \operatorname{rk}^G(u^{\boxtimes n}) \leq \operatorname{srk}(u^{\boxtimes n})$. Using the G-stable rank, we get better upper bounds for the cardinality of a cap set (or a tricolored sum-free set). Below is a table of the upper bounds we get for $n \leq 20$. We compared our bound to the bound of Ellenberg and Gijswijt that is based on the slice rank. In the comment section of Tao's blog [2016], Fedor Petrov outlined a more refined argument to improve the upper bound for the cardinality of cap sets. We also compared our bounds with Petrov's bound. The comparisons are given in Table 1.

As we see, our bounds improve the bounds of Ellenberg and Gijswijt, but not the bounds of Petrov. Since Petrov's argument uses the symmetry, it is not clear whether his bound is also an upper bound for the tricolored sum-free sets. Also, this bound does not exactly come from bounds for the slice rank, but may be related to some other notion of rank. It would be interesting to see if the notion of G-stable rank could be combined with Petrov's approach to obtain even sharper bounds for the cap set problem.

Since the slice rank and the G-stable rank are the same up to a constant, the asymptotic slice rank and the asymptotic G-stable rank are the same. It was shown in [Christandl et al. 2018] that, over the complex numbers, the asymptotic slice rank can be expressed in quantum functionals. It was also noted there that the Ellenberg–Gijswijt bound for the cap set problem is closely related to the Strassen's computation of the asymptotic spectrum of the multiplication tensor of the algebra $\mathbb{F}_3[x]/(x^3)$; see [Strassen 1991].

2. The G-stable rank and the Hilbert–Mumford criterion

2A. ***The Hilbert–Mumford criterion.*** We will discuss the K-rational version of the Hilbert–Mumford criterion by Kempf [1978]. We remind the reader that the base field K is assumed to be perfect. Suppose that G is a connected reductive algebraic group over a field K, X is a separated K-scheme of finite type and $G \times X \to X$ is a G-action that is also a morphism of schemes over K. The multiplicative group is defined as $\mathbb{G}_m = \operatorname{Spec} K[t, t^{-1}]$. A 1-parameter subgroup of G is a homomorphism $\lambda : \mathbb{G}_m \to G$ of algebraic groups. We say that this 1-parameter subgroup of G is K-rational if the homomorphism is a morphism of algebraic varieties defined over K. In the case where K is finite, we caution the reader that the set $G(K)$ of K rational points in G is finite and may not be Zariski dense in the algebraic group G. If $x \in X(K)$ is a K-rational point of X, then $G \cdot x$ denotes a subscheme of X which is not necessarily Zariski closed (even if $G(K)$ is finite). The Zariski closure $\overline{G \cdot x}$ is a closed subscheme of X.

Theorem 2.1 [Kempf 1978, Corollary 4.3]. *Suppose that $x \in X(K)$ is a K-rational point, $S \subseteq X$ is a G-invariant closed subscheme of X such that $\overline{G \cdot x} \cap S \neq \varnothing$, Then there exists a K-rational 1-parameter subgroup $\lambda : \mathbb{G}_m \to G$ such that $\lim_{t \to 0} \lambda(t) \cdot x = y$ for some $y \in S(K)$.*

In our situation, $X = V$ is a K-vector space which is a representation of G, and $S = \{0\}$. A vector $v \in V$ is called G-semistable if $\overline{G \cdot v}$ does not contain 0. Now Theorem 2.1 implies:

Corollary 2.2. *If G is a connected reductive algebraic group, $v \in V$ and $0 \in \overline{G \cdot v}$ then there exists a K-rational 1-parameter subgroup $\lambda : \mathbb{G}_m \to G$ such that $\lim_{t \to 0} \lambda(t) \cdot v = 0$.*

A 1-parameter subgroup of GL_n is of the form

$$\lambda(t) = C \begin{pmatrix} t^{x(1)} & & & \\ & t^{x(2)} & & \\ & & \ddots & \\ & & & t^{x(n)} \end{pmatrix} C^{-1}$$

with $C \in \mathrm{GL}_n$ and $x(1), x(2), \dots, x(n) \in \mathbb{Z}$. In particular, we can view λ as an element of $\mathrm{GL}_n(K[t, t^{-1}])$ where $K[t, t^{-1}] \subseteq K((t))$ is the ring of Laurent polynomials. If $v = (v_1\ v_2\ \cdots\ v_n)^t \in K^n$ then $\lim_{t \to 0} \lambda(t) \cdot v = 0$ if for all i, we have $v_i = 0$ or $x(i) > 0$. We will take $V = V_1 \otimes V_2 \otimes \cdots \otimes V_d$ and $G = \mathrm{GL}(V_1) \times \mathrm{GL}(V_2) \times \cdots \times \mathrm{GL}(V_d)$. A 1-parameter subgroup of G is of the form $(\lambda_1(t), \lambda_2(t), \dots, \lambda_d(t))$ where $\lambda_i(t) : \mathbb{G}_m \to \mathrm{GL}(V_i)$ is a 1-parameter subgroup for all i.

For an integer vector $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d) \in \mathbb{Z}^d$ we define a homomorphism of algebraic groups $\det^\alpha : G \to \mathbb{G}_m$ by $(A_1, \dots, A_d) \mapsto \prod_{i=1}^d \det(A_i)^{\alpha_i}$. This homomorphism corresponds to a 1-dimensional representation of G, which we will also denote by $\det^\alpha$. We will now relate the G-stable rank to semistability in geometric invariant theory. We compare the G-stable rank with a rational number p/q and for this we use semistability in the representations $V^{\otimes p}$ and certain twists with products of determinants.

Proposition 2.3. *Suppose that $\beta \in \mathbb{Q}^d_{>0}$, p is a nonnegative integer and q is a positive integer with $q\beta \in \mathbb{Z}^n$. We define a representation W by*

$$W = (V^{\otimes p} \otimes \det{}^{-q\beta}) \oplus V_1^{n_1} \oplus V_2^{n_2} \oplus \cdots \oplus V_d^{n_d}.$$

and choose $u_i \in V_i^{n_i} \cong K^{n_i \times n_i}$ of maximal rank n_i for every i. Then we have $\mathrm{rk}^G_\beta(v) \geq p/q$ if and only if $w = (v^{\otimes p} \otimes 1, u_1, \dots, u_d)$ is G-semistable.

Proof. Suppose that $\mathrm{rk}^G_\beta(v) < p/q$. Then there exists $g(t) = (g_1(t), \dots, g_d(t)) \in G(K[\![t]\!])$ with

$$\mathrm{val}_t(g(t) \cdot (v^{\otimes p} \otimes 1)) = p\, \mathrm{val}_t(g(t) \cdot v) - \sum_{i=1}^d q\beta_i\, \mathrm{val}_t(g_i(t)) > 0.$$

The limit $\lim_{t\to 0} g(t)\cdot w = (0, g(0)\cdot u) = (0, g(0)\cdot u_1, \dots, g(0)\cdot u_d)$ lies in the closure of the orbit $G\cdot w$. Since 0 lies in the orbit closure of $(0, g(0)\cdot u)$, it also lies in the orbit closure of w. We conclude that w is not G-semistable.

Now suppose that w is not G-semistable. By the Hilbert–Mumford criterion, there exists a 1-parameter subgroup $\lambda(t) = (\lambda_1(t), \dots, \lambda_d(t)) \in G(K[t, t^{-1}])$ of G such that $\lim_{t\to 0} \lambda(t)\cdot w = 0$. This implies that $\lim_{t\to 0} \lambda_i(t)\cdot u_i = 0$. Since u_i has maximal rank, we get $\lim_{t\to 0} \lambda_i(t) = 0$ and $\lambda_i(t) \in \mathrm{GL}(V_i, K[t])$. So we have $\lambda(t) \in G(K[t]) \subseteq G(K[\![t]\!])$. We also get

$$0 < \mathrm{val}_t(\lambda(t)\cdot(v^{\otimes p}\otimes 1)) = p\,\mathrm{val}_t(\lambda(t)\cdot v) - \sum_{i=1}^{d} q\beta_i\,\mathrm{val}_t(\lambda_i(t))$$

and therefore

$$\mu_\beta^G(v) = \frac{\sum_{i=1}^d \beta_i\,\mathrm{val}_t(\lambda_i(t))}{\mathrm{val}_t(\lambda(t)\cdot v} < \frac{p}{q}.$$

We conclude that $\mathrm{rk}_\beta^G(v) < p/q$. ☐

Theorem 2.4. *If $\alpha \in \mathbb{R}^d_{>0}$, then the G-stable rank $\mathrm{rk}_\alpha^G(v)$ is the infimum of $\mu_\alpha(\lambda(t), v)$ where $\lambda(t) \in G(K[t])$ is a 1-parameter subgroup of G and $\mathrm{val}_t(\lambda(t)\cdot v) > 0$.*

Proof. Assume that $\mathrm{rk}_\alpha^G(v) < r$ for some rational number r. There exists a $\beta \in \mathbb{Q}^d_{>0}$ with $\beta - \alpha \in \mathbb{R}^d_{>0}$ and $\mathrm{rk}_\beta^G(v) < r$. We can write $r = p/q$ where p and q are positive integers such that $q\beta \in \mathbb{Z}^d$. By Proposition 2.3, w is not G-semistable and from the proof of Proposition 2.3 follow that there exists a 1-parameter subgroup $\lambda(t) \in G(K[t])$ such that $\mu_\alpha(\lambda(t), v) \le \mu_\beta(\lambda(t), v) < r$. This shows that even if $\lambda(t) \in G(K[t])$ is a 1-parameter subgroup of G, $\mu_\alpha(\lambda(t), v)$ can get arbitrarily close to $\mathrm{rk}_\alpha^G(v)$. ☐

2B. ***The relation between G-stable rank and* SL*-stability.*** First we prove that the G-stable rank does not change when we extend the field.

Theorem 2.5. *Suppose that $v \in V = V_1 \otimes_K V_2 \otimes_K \otimes \cdots \otimes_K V_d$ where $V_1, V_2, \dots, V_d$ are finite dimensional K-vector spaces, and $\bar{v} = 1 \otimes v \in \overline{V} = L \otimes_K V \cong \overline{V}_1 \otimes_L \overline{V}_2 \otimes_L \otimes \cdots \otimes_L \overline{V}_d$ with $\overline{V}_i = L \otimes_K V_i$ for all i. Then we have $\mathrm{rk}_\alpha^G(v) = \mathrm{rk}_\alpha^G(\bar{v})$. In other words, the G-stable rank does not change under base field extension.*

Proof. If $\beta \in \mathbb{Q}^d_{>0}$ then we can follow the set up in Proposition 2.3, where $p, q \in \mathbb{Z}$, $p \ge 0$, $q > 0$ and $q\beta \in \mathbb{Z}^d$. We choose $u_i \in V_i^{n_i}$ invertible for all i, and define

$$w = (v^{\otimes p}\otimes 1, u_1, \dots, u_d) \in W = (V^{\otimes p}\otimes_K \det{}^{-q\beta}) \oplus V_1^{n_1} \oplus V_2^{n_2} \oplus \cdots \oplus V_d^{n_d}.$$

Using the base field extension, we get

$$\bar{w} = (\bar{v}\otimes 1, \bar{u}_1, \dots, \bar{u}_d) \in L\otimes_K W = (\overline{V}^{\otimes p}\otimes_L \det{}^{-q\beta}) \oplus \overline{V}_1^{n_1} \oplus \overline{V}_2^{n_2} \oplus \cdots \oplus \overline{V}_d^{n_d}.$$

Now G-semistability does not chance after base field extension. So w is G-semistable if and only if $\bar{w}$ is G-semistable. So we have

$$\mathrm{rk}^G_\beta(w) \geq \tfrac{p}{q} \Leftrightarrow w \text{ is } G\text{-semistable} \Leftrightarrow \bar{w} \text{ is } G\text{-semistable} \Leftrightarrow \mathrm{rk}^G_\beta(\bar{w}) \geq \tfrac{p}{q}.$$

This proves that $\mathrm{rk}^G_\beta(w) = \mathrm{rk}^G_\beta(\bar{w})$. Since $\mathrm{rk}^G_\alpha(w)$ is the supremum of $\mathrm{rk}^G_\beta(w)$ over all $\beta \in \mathbb{Q}^d_{>0}$ with $\beta \leq \alpha$, we also get $\mathrm{rk}^G_\alpha(w) = \mathrm{rk}^G_\alpha(w)$ for all $\alpha \in \mathbb{R}^d_{>0}$. □

Proposition 2.6. *Suppose that $\alpha = (1/n_1, 1/n_2, \ldots, 1/n_d)$ where $n_i = \dim V_i$. For $v \in V = V_1 \otimes V_2 \otimes \cdots \otimes V_d$ we have $\mathrm{rk}^G_\alpha(v) \leq 1$. Moreover, $\mathrm{rk}^G_\alpha(v) = 1$ if and only if v is semistable with respect to the group $H = \mathrm{SL}(V_1) \times \mathrm{SL}(V_2) \times \cdots \times \mathrm{SL}(V_d)$.*

Proof. The inequality $\mathrm{rk}^G_\alpha(v) \leq 1$ is obvious. Suppose that $v \in V$ is not H-semistable. Then there exists a 1-parameter subgroup $\lambda(t) = (\lambda_1(t), \ldots, \lambda_d(t)) : \mathbb{G}_m \to H$ with $\lim_{t\to 0} \lambda(t) \cdot v = 0$. We can choose $c_1, c_2, \ldots, c_d$ such that $\lambda'(t) = (t^{c_1}\lambda_1(t), \ldots, t^{c_d}\lambda_d(t)) \in G(K[t])$. Note that $\det(t^{c_i}\lambda_i(t)) = \det(t^{c_i} I_{n_i}) \det(\lambda_i(t)) = t^{c_i n_i}$. Now we have $\mathrm{val}_t(\lambda'(t) \cdot v) = s + c_1 + c_2 + \cdots + c_d$ and

$$\mu(\lambda'(t), v) = \frac{\sum_{i=1}^d \frac{1}{n_i} \mathrm{val}_t(\det(t^{c_i}\lambda_i(t)))}{\mathrm{val}_t(\lambda'(t) \cdot v)} = \frac{\sum_{i=1}^d c_i}{s + \sum_{i=1}^d c_i} < 1.$$

This proves that $\mathrm{rk}^G_\alpha(v) < 1$.

Conversely, suppose that $\mathrm{rk}^G_\alpha(v) < 1$. Choose a polynomial 1-parameter subgroup of G such that $\mathrm{val}_t(\lambda(t) \cdot v) = s > 0$ and $\mu_\alpha(\lambda(t), v) < 1$. Let $c_i = \mathrm{val}_t(\det \lambda_i(t))$. Then we have $\mu_\alpha(\lambda(t), v) = \sum_{i=1}^d c_i/n_i < s$. After replacing t by t^k for some positive integer k we may assume that $c_i/n_i \in \mathbb{Z}$ for all i. Let $\lambda'(t) = (t^{-c_1/n_1}\lambda_1(t), t^{-c_2/n_2}\lambda_2(t), \ldots, t^{-c_d/n_d}\lambda_d(t))$. Then $\lambda'(t)$ is a 1-parameter subgroup of H and $\mathrm{val}_t(\lambda'(t) \cdot v) = s - \sum_{i=1}^d c_i/n_i > 0$, so $\lim_{t\to 0} \lambda'(t) \cdot v = 0$. This shows that v is H-unstable. □

2C. *The G-stable rank and the noncommutative rank.* The noncommutative rank is defined as the rank of $A(t) = t_1 A_1 + t_2 A_2 + \cdots + t_m A_m$ where $t_1, t_2, \ldots, t_m$ are variables in the free skew field $R = K\langle\!\langle t_1, t_2, \ldots, t_m \rangle\!\rangle$ and $A(t)$ is viewed as a $p \times q$ matrix with entries in R; see [Fortin and Reutenauer 2004; Cohn 1995] for more on free skew fields. We will use the following equivalent definition; see [Fortin and Reutenauer 2004]:

Definition 2.7. Suppose that $A_1, A_2, \ldots, A_m$ are $p \times q$ matrices. Then the noncommutative rank $\mathrm{ncrk}(A)$ of $A = (A_1, \ldots, A_m)$ is equal to the maximal value of

$$q + \dim \sum_{i=1}^m A_i(W) - \dim W$$

over all subspaces $W \subseteq K^q$.

It was shown in [Ivanyos et al. 2017] that the noncommutative rank of A is also equal to maximum of

$$\frac{\mathrm{rk}\left(\sum_{i=1}^m T_i \boxtimes A_i\right)}{d}$$

where d is a positive integer, $T_1, T_2, \dots, T_m$ are $d \times d$ matrices, and $\boxtimes$ is the Kronecker product of two matrices (so $T_i \boxtimes A_i$ is a $dp \times dq$-matrix).

The noncommutative rank relates to stability. If A is an m-tuple of $n \times n$ matrices (i.e., $p = q = n$) then $\operatorname{ncrk}(A) = n$ if and only if A is semistable with respect to the simultaneous left-right action of $\mathrm{SL}_n \times \mathrm{SL}_n$ on m-tuples of matrices; see [Ivanyos et al. 2017].

We can relate the noncommutative and G-stable rank as follows. First, we will view the m-tuple $A = (A_1, A_2, \dots, A_m)$ as a tensor. Using a linear isomorphism $K^p \otimes K^q \cong K^{p \times q}$, we can view $A_1, A_2, \dots, A_m$ as tensors in $K^p \otimes K^q$. The m-tuple $A = (A_1, A_2, \dots, A_m)$ corresponds to a tensor $T_A = \sum_{i=1}^m A_i \otimes [i] \in K^p \otimes K^q \otimes K^m$.

Lemma 2.8. *The noncommutative rank is the smallest value of $r + s$ for which there exist linearly independent vectors $v_1, \dots, v_r \in K^p$ and linearly independent vectors $w_1, \dots, w_s \in K^q$ with*

$$T_A \in \sum_{i=1}^r v_i \otimes K^q \otimes K^m + \sum_{j=1}^s K^p \otimes w_j \otimes K^m. \tag{3}$$

Proof. If (3) holds, then take W to be the $(q-s)$-dimensional space perpendicular to the vectors $w_1, w_2, \dots, w_s$. The space $A_i(W)$ is contained in the span of $v_1, v_2 \dots, v_r$. So the noncommutative rank is at most $q + r - (q - s) = r + s$.

We show that $r + s$ can be equal to $\operatorname{ncrk}(A)$. Suppose that $k = \operatorname{ncrk}(A)$. For some s there exists an subspace $V \subseteq K^p$ with $k = q + \dim V - \dim W$, where $V = \sum_{i=1}^m A_i(W)$. Choose a basis $w_1, w_2, \dots, w_s$ of the space orthogonal to W. Then we have $s = q - \dim W$. Also choose a basis $v_1, v_2, \dots, v_r$ of V. Now (3) holds and $r + s = q - \dim W + \dim V = k$. □

The following proposition shows that the noncommutative rank can be seen as a special case of the G-stable rank.

Proposition 2.9. *For $\alpha = (1, 1, \ell)$ and $\ell \geq \min\{p, q\}$ we have* $\operatorname{ncrk}(A) = \operatorname{rk}^G_\alpha(T_A)$.

Proof. Let $k = \operatorname{ncrk}(A)$. Then we have

$$T_A \in \sum_{i=1}^r v_i \otimes K^q \otimes K^m + \sum_{j=1}^s K^p \otimes w_j \otimes K^m,$$

for some r and s with $r + s = k$ and vectors $v_1, \dots, v_r, w_1, \dots, w_s$. We extend $v_1, \dots, v_r$ to a basis $v_1, \dots, v_p$ and extend $w_1, \dots, w_s$ to a basis $w_1, \dots, w_q$. We define a 1-parameter subgroup $\lambda(t) = (\lambda_1(t), \lambda_2(t), \lambda_3(t))$ in $G = \mathrm{GL}_p \times \mathrm{GL}_q \times \mathrm{GL}_m$ by $\lambda_1(t) \cdot v_i = t v_i$ for $i = 1, 2, \dots, r$, $\lambda_1(t) \cdot v_i = v_i$ for $i = r+1, r+2, \dots, p$, $\lambda_2(t) \cdot w_j = t w_j$ for $j = 1, 2, \dots, s$, $\lambda_2(t) \cdot w_j = w_j$ for $j = s+1, s+2, \dots, q$ and $\lambda_3(t)$ is just the identity. Then we have $\operatorname{val}_t(\lambda(t) \cdot T_A) = 1$, $\det(\lambda_1(t)) = t^r$, $\det(\lambda_2(t)) = t^s$, $\det(\lambda_3(t)) = 1$ and

$$\operatorname{rk}^G_\alpha(T_A) \leq \mu_\alpha(\lambda(t), T_A) = \frac{1 \cdot r + 1 \cdot s + \ell \cdot 0}{1} = k = \operatorname{ncrk}(A).$$

On the other hand, let $h = \mathrm{rk}^G_\alpha(T_A)$ and suppose that $\lambda(t) \in G$ is a 1-parameter subgroup with $\mu_\alpha(\lambda(t), T_A) = h$. If $h = \min\{p, q\}$ then clearly $\mathrm{ncrk}(A) \leq h$, so we assume that $h < \min\{p, q\}$. Suppose $\ell \geq p$ (the case $\ell \geq q$ will go similarly). If $\det(\lambda_3(t)) = t^e$ then we can define another 1-parameter subgroup $\rho(t) = (\rho_1(t), \rho_2(t), \rho_3(t))$ by $\rho_1(t) = t^e\lambda_1(t)$, $\rho_2(t) = \lambda_2(t)$ and $\rho_3(t) = I$. Then $\mathrm{val}_t(\rho(t)\cdot T_A) \geq \mathrm{val}_t(\lambda(t)\cdot T_A)$, and we get

$$\begin{aligned}\mu_\alpha(\rho(t), T_A) &= \frac{\mathrm{val}_t(\det\rho_1(t)) + \mathrm{val}_t(\det\rho_2(t)) + \ell\,\mathrm{val}_t(\det\rho_3(t))}{\mathrm{val}_t(\rho(t)\cdot T_A)}\\ &\leq \frac{pe + \mathrm{val}_t(\det\lambda_1(t)) + \mathrm{val}_t\det(\lambda_2(t))}{\mathrm{val}_t(\lambda(t)\cdot T_A)}\\ &\leq \frac{\mathrm{val}_t(\det\lambda_1(t)) + \mathrm{val}_t(\det\lambda_2(t)) + \ell\,\mathrm{val}_t(\det\lambda_3(t))}{\mathrm{val}_t(\lambda(t)\cdot T_A)}\\ &= \mu_\alpha(\lambda(t), T_A)\end{aligned}$$

because $\ell \geq p$ and $\mathrm{val}_t(\det\lambda_3(t)) = e$. We can replace $\lambda(t)$ by $\rho(t)$ and without loss of generality we may assume that $\lambda_3(t) = I$.

Let $d := \mathrm{val}_t(\lambda(t)\cdot T_A)$. After base changes, we have

$$\lambda(t) = \begin{pmatrix} t^{x(1)} & & \\ & \ddots & \\ & & t^{x(p)}\end{pmatrix} \quad\text{and}\quad \rho(t) = \begin{pmatrix} t^{y(1)} & & \\ & \ddots & \\ & & t^{y(q)}\end{pmatrix}.$$

From

$$\frac{\sum_{i=1}^{h+1}(x(i) + y(h+2-i))}{d} \leq \frac{\sum_{i=1}^{p} x(i) + \sum_{j=1}^{q} y(j)}{d} = \mu_\alpha(\lambda(t), T_A) = h$$

follows that $x(r+1) + y(s+1) \leq hd/(k+1) < hd$ for some r, s with $r + s = h$. If a basis vector $[i, j, k] = [i]\otimes[j]\otimes[k]$ appears in T_A then $x(i) + y(j) \geq dk$ and therefore $i \leq r$ or $j \leq s$. This means that

$$T_A \in \sum_{i=1}^{r}[i]\otimes K^q \otimes K^m + \sum_{j=1}^{s} K^p \otimes [j] \otimes K^m$$

and $\mathrm{ncrk}(T_A) \leq r + s = h = \mathrm{rk}^G_\alpha(T_A)$. □

2D. ***Semicontinuity of the G-stable rank.*** We will show that the G-stable rank is semicontinuous, which means that for every r, the set of all tensors with G-stable rank $\leq r$ is Zariski closed.

Let us for the moment fix a 1-parameter subgroup $\lambda(t)$ of G. We can choose bases in the vector spaces V_i for $i = 1, 2, \ldots, d$ such that the matrix of $\lambda_i(t)$ is diagonal, with diagonal entries $t^{x(i,1)}, t^{x(i,2)}, \ldots, t^{x(i,n_i)}$ where $x(i, 1) \geq x(i, 2) \geq \cdots \geq x(i, n_i) \geq 0$. Define

$$Z = \{v \in V \mid \mu_\alpha(\lambda(t), v) < r\}.$$

The space Z is spanned by all basis vectors $[i_1, i_2, \dots, i_d] \in V$ with

$$\sum_{i=1}^{d} \alpha_i \sum_{j=1}^{n_i} x(i, j) < r(x(1, i_1) + x(2, i_2) + \cdots + x(d, i_d)).$$

Let $B = B_{n_1} \times B_{n_2} \times \cdots \times B_{n_d} \subseteq G$ where $B_k \subseteq \mathrm{GL}_k$ is the Borel group of upper triangular invertible matrices. If $[i_1, i_2, \dots, i_d]$ lies in Z, and $j_k \leq i_k$ for all k, then $[j_1, j_2, \dots, j_d]$ lies in Z. This implies that Z is stable under the action of B.

Lemma 2.10. *The set $G \cdot Z = \bigcup_{g \in G} g \cdot Z$ is Zariski closed.*

Proof. Consider the Zariski closed subset $S \subseteq G/B \times V$ defined by

$$S = \{(gB, v) \mid g^{-1} \cdot v \in Z\}$$

and let $\pi : G/B \times V \to V$ be the projection onto V. The flag variety G/B is projective, so π is a projective morphism which maps closed sets to closed sets. In particular, $G \cdot Z = \pi(S)$ is Zariski closed. □

Theorem 2.11. *For any weight $\alpha \in \mathbb{R}^d_{>0}$ and $r \in \mathbb{R}$ the sets $X^\circ(\mathrm{rk}^G_\alpha, r) = \{v \in V \mid \mathrm{rk}^G_\alpha(v) < r\}$ and $X(\mathrm{rk}^G_\alpha, r) = \{v \in V \mid \mathrm{rk}^G_\alpha(v) \leq r\}$ are finite unions of sets of the form $G \cdot Z$ where Z is a Borel-fixed subspace. In particular, these sets are Zariski closed.*

Proof. If $\mathrm{rk}^G_\alpha(v) < r$, then there exists a 1-parameter subgroup $\lambda(t)$ of G such that $\mu_\alpha(\lambda(t), v) < r$. If $Z = \{w \in V \mid \mu_\alpha(\lambda(t), w) < r\}$ then $X^\circ(\mathrm{rk}^G_\alpha, r)$ contains Z and $G \cdot Z$. Since there are only finite many Borel stable subspaces of V, we see that $X^\circ(\mathrm{rk}^G_\alpha, r)$ must be a finite union $G \cdot Z_1 \cup G \cdot Z_2 \cup \cdots \cup G \cdot Z_s$ where $Z_1, Z_2, \dots, Z_s$ are Borel stable subspaces. Since each $G \cdot Z_i$ is closed, $X^\circ(\mathrm{rk}^G_\alpha, r)$ is closed. Because there are only finitely many Borel stable subspaces, there are only finitely many possibilities for $X^\circ(\mathrm{rk}^G_\alpha, s)$ where $s \in \mathbb{R}_{>0}$. There exists an $\varepsilon > 0$ such that $X^\circ(\mathrm{rk}^G_\alpha, s)$ is the same for all $s \in (r, r + \varepsilon]$. We have $X(\mathrm{rk}^G_\alpha, r) = \bigcap_{r < s \leq r+\varepsilon} X^\circ(\mathrm{rk}^G_\alpha, s) = X^\circ(\mathrm{rk}^G_\alpha, r + \varepsilon)$. □

3. Results on the G-stable rank

3A. *Easy observations and a technical lemma.*

Lemma 3.1. *If $v \neq 0$, then we have $\mathrm{rk}^G_\alpha(v) \geq \min\{\alpha_1, \alpha_2, \dots, \alpha_d\} > 0$. In particular, $\mathrm{rk}^G(v) \geq 1$.*

Proof. Choose $g(t) \in G(K[\![t]\!])$ with $\mu_\alpha(g(t), v) = \mathrm{rk}^G_\alpha(v)$. From $v \neq 0$ follows that $g(t) \cdot v \neq 0$, say $\mathrm{val}_t(g(t) \cdot v) = s > 0$. Then we get $\sum_{i=1}^d \mathrm{val}_t(g_i(t)) \geq s$ and

$$\frac{\sum_{i=1}^d \alpha_i \,\mathrm{val}_t(g_i(t))}{\mathrm{val}_t(g(t) \cdot v)} \geq \min\{\alpha_1, \dots, \alpha_d\} \frac{\sum_{i=1}^s \mathrm{val}_t(g_i(t))}{s} \geq \min\{\alpha_1, \dots, \alpha_d\}.$$

It follows that $\mathrm{rk}^G_\alpha(v) \geq \min\{\alpha_1, \dots, \alpha_d\} > 0$. □

Suppose that $v = u \otimes w$ is nonzero with $u \in V_1$ and $w \in V_2 \otimes \cdots \otimes V_d$. We choose bases in $V_1, \ldots, V_d$ such that u is the first basis vector in V_1. We can choose a one parameter subgroup $\lambda(t)$ with

$$\lambda_1(t) = \begin{pmatrix} t & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

and $\lambda_k(t) = 1_{n_k}$ for $k = 2, 3, \ldots, d$. Then we have $\lambda(t) \cdot v = tv$ and $\mu_\alpha(A(t), v) = \alpha_1$. This shows that $\mathrm{rk}^G_\alpha(v) \le \alpha_1$. From Lemma 3.1 follows that $\mathrm{rk}^G_\alpha(v) \le \alpha_1$. If v has slice rank 1 concentrated in the i-th slice, then $\mathrm{rk}^G_\alpha(v) \le \alpha_i \le \max\{\alpha_1, \alpha_2, \ldots, \alpha_d\}$.

Corollary 3.2. *If v has slice rank* 1, *then* $\mathrm{rk}^G(v) = 1$.

Proof. If v has slice rank 1, $\mathrm{rk}^G(v) = \mathrm{rk}^G_{(1,\ldots,1)}(v) \le \max\{1, \ldots, 1\} = 1$ and $\mathrm{rk}^G(v) \ge 1$ by Lemma 3.1. □

Corollary 3.3. *If v has rank* 1 *then* $\mathrm{rk}^G_\alpha(v) = \min\{\alpha_1, \ldots, \alpha_d\}$.

Proof. If v has rank 1 then $\mathrm{rk}^G_\alpha \le \alpha_i$ for every i and $\mathrm{rk}^G_\alpha \ge \min\{\alpha_1, \ldots, \alpha_d\}$ by Lemma 3.1. □

Proposition 3.4. *Suppose that $v \in V_1 \otimes V_2 \otimes \cdots \otimes V_d$ and $w \in W_1 \otimes W_2 \otimes \cdots \otimes W_e$ and $v \otimes w \in V_1 \otimes \cdots \otimes V_d \otimes W_1 \otimes \cdots \otimes W_e$ is the horizontal tensor product. We have* $\mathrm{rk}^G_{\alpha,\beta}(v \otimes w) = \min\{\mathrm{rk}^G_\alpha(v), \mathrm{rk}^G_\beta(w)\}$.

Proof. Let $G = \mathrm{GL}(V_1) \times \cdots \times \mathrm{GL}(V_d)$ and $H = \mathrm{GL}(W_1) \times \cdots \times \mathrm{GL}(W_e)$. There exists $g(t) \in G(K[\![t]\!])$ with $\mu_\alpha(g(t), v) = \mathrm{rk}_\alpha(v)$. For $(g(t), 1) \in (G \times H)(K[\![t]\!])$ we get $\mu_{\alpha,\beta}((g(t), h(t)), v \otimes w) = \mathrm{rk}_\alpha(v)$. This proves that $\mathrm{rk}^G_{\alpha,\beta}(v \otimes w) \le \mathrm{rk}_\alpha(v)$. Similarly, we have $\mathrm{rk}^G_{\alpha,\beta}(v \otimes w) \le \mathrm{rk}^G_\beta(w)$, so we get $\mathrm{rk}^G_{\alpha,\beta}(v \otimes w) \le \min\{\mathrm{rk}^G_\alpha(v), \mathrm{rk}^G_\beta(w)\}$.

Conversely, suppose that $(g(t), h(t)) \in G \times H(K[\![t]\!])$ satisfies $\mu_{\alpha,\beta}((g(t), h(t)), v \otimes w) = \mathrm{rk}^G_{\alpha,\beta}(v \otimes w)$. Using that

$$\mathrm{val}_t((g(t), h(t)) \cdot (v \otimes w)) = \mathrm{val}_t((g(t) \cdot v) \otimes (h(t) \cdot w)) = \mathrm{val}_t(g(t) \cdot v) + \mathrm{val}_t(h(t) \cdot w)$$

we get

$$\begin{aligned} \mu_{\alpha,\beta}(v \otimes w) &= \frac{\sum_{i=1}^d \mathrm{val}_t(\det g_i(t)) + \sum_{j=1}^e \mathrm{val}_t(\det h_j(t))}{\mathrm{val}_t(g(t) \cdot v) + \mathrm{val}_t(h(t) \cdot w)} \\ &\ge \min\left\{ \frac{\sum_{i=1}^d \mathrm{val}_t(\det g_i(t))}{\mathrm{val}_t(g(t) \cdot v)}, \frac{\sum_{j=1}^e \mathrm{val}_t(\det h_j(t))}{\mathrm{val}_t(h(t) \cdot w)} \right\} \\ &= \min\{\mathrm{rk}^G_\alpha(v), \mathrm{rk}^G_\beta(w)\}. \end{aligned}$$

□

We will need the following technical lemma to prove Proposition 3.6.

Lemma 3.5. *If $g(t), h(t) \in \mathrm{GL}_n(K[\![t]\!])$ then there exists $u(t), g'(t), h'(t) \in \mathrm{GL}_n(K[\![t]\!])$ such that $u(t) = g'(t)h(t) = h'(t)g(t)$ and* $\mathrm{val}_t(\det u(t)) \le \mathrm{val}_t(\det g(t)) + \mathrm{val}_t(\det h(t))$.

Proof. We have

$$\mathrm{val}_t(\det g(t)) = \dim_K \frac{K[\![t]\!]^n}{g(t) K[\![t]\!]^n}.$$

The $K[\![t]\!]$-module $g(t)K[\![t]\!]^n \cap h(t)K[\![t]\!]^n$ is a submodule of the free module $K[\![t]\!]^n$, so it is also free of rank $\leq n$. So there exists a matrix $u(t)$ such that $g(t)K[\![t]\!]^n \cap h(t)K[\![t]\!]^n = u(t)K[\![t]\!]^n$. From $u(t)K[\![t]\!]^n \subseteq g(t)K[\![t]\!]^n$ follows that there exists a matrix $h'(t)$ such that $u(t) = h'(t)g(t)$. Similarly, we find a matrix $g'(t)$ with $u(t) = g'(t)h(t)$.

We have

$$\begin{aligned}
\mathrm{val}_t(\det u(t)) \leq \dim \frac{K[\![t]\!]^n}{u(t)K[\![t]\!]^n} &= \dim \frac{K[\![t]\!]^n}{g(t)K[\![t]\!]^n \cap h(t)K[\![t]\!]^n} \\
&= \dim \frac{K[\![t]\!]^n}{g(t)K[\![t]\!]^n} + \dim \frac{g(t)K[\![t]\!]^n}{g(t)K[\![t]\!]^n \cap h(t)K[\![t]\!]^n} \\
&= \mathrm{val}_t(\det g(t)) + \dim \frac{g(t)K[\![t]\!]^n + h(t)K[\![t]\!]^n}{h(t)K[\![t]\!]^n} \\
&\leq \mathrm{val}_t(\det g(t)) + \mathrm{val}_t(\det h(t)). \qquad \square
\end{aligned}$$

3B. *The triangle inequality for the G-stable rank.*

Proposition 3.6. *For tensors $v, w \in V$ we have $\mathrm{rk}^G_\alpha(v + w) \leq \mathrm{rk}^G_\alpha(v) + \mathrm{rk}^G_\alpha(w)$.*

Proof. Suppose that $g(t), h(t) \in G(K[\![t]\!])$. If we replace t by t^e, then $\mu_\alpha(g(t), v)$ does not change. Without changing $\mu_\alpha(g(t), v)$ and $\mu_\alpha(h(t), w)$ we may assume that $\mathrm{val}_t(g(t) \cdot v) = \mathrm{val}_t(h(t) \cdot w) = s > 0$. Then there exist $u(t), g'(t), h'(t) \in G(K[\![t]\!])$ such that $u(t) = h'(t)g(t) = g'(t)h(t)$ and $\mathrm{val}_t(\det u_i(t)) \leq \mathrm{val}_t(\det g_i(t)) + \mathrm{val}_t(\det h_i(t))$ for all i by Lemma 3.5. We get

$$\begin{aligned}
\mathrm{val}_t(u(t) \cdot (v + w)) &= \mathrm{val}_t(h'(t)g(t) \cdot v + g'(t)h(t) \cdot w) \\
&\geq \min\{\mathrm{val}_t(h'(t)g(t) \cdot v), \mathrm{val}_t(g'(t)h(t) \cdot w\} \\
&\geq \min\{\mathrm{val}_t(g(t) \cdot v), \mathrm{val}_t(h(t) \cdot w)\} \\
&= s
\end{aligned}$$

and

$$\sum_{i=1}^d \alpha_i \,\mathrm{val}_t(\det u_i(t)) \leq \sum_{i=1}^d \alpha_i \,\mathrm{val}_t(\det g_i(t)) + \sum_{i=1}^d \alpha_i \,\mathrm{val}_t(\det h_i(t)) = s\mu_\alpha(g(t), v) + s\mu_\alpha(h(t), w).$$

It follows that

$$\mu_\alpha(u(t), v+w) = \frac{\sum_{i=1}^d \alpha_i \,\mathrm{val}_t(\det u_i(t))}{\mathrm{val}_t(u(t) \cdot (v + w))} \leq \frac{s\mu_\alpha(g(t), v) + s\mu_\alpha(h(t), w)}{s} = \mu_\alpha(g(t), v) + \mu_\alpha(h(t), w).$$

Taking the infimum over all $g(t)$ and $h(t)$ gives $\mathrm{rk}^G_\alpha(v + w) \leq \mathrm{rk}^G_\alpha(v) + \mathrm{rk}^G_\alpha(w)$. $\square$

Corollary 3.7. *For any tensor $v \in V$ we have*

$$\mathrm{rk}^G(v) \leq \mathrm{srk}(v).$$

Proof. By definition, we can write $v = v_1 + v_2 + \cdots + v_r$ where $r = \mathrm{srk}(v)$ and $v_1, v_2, \ldots, v_r$ are tensors of slice rank 1. Now $\mathrm{rk}^G(v) = \mathrm{rk}^G(v_1 + \cdots + v_r) \leq \mathrm{rk}^G(v_1) + \cdots + \mathrm{rk}^G(v_r) = 1 + \cdots + 1 = r = \mathrm{srk}(v)$. $\square$

3C. ***The additive property of the G-stable rank.***

Proposition 3.8. *If $d \geq 2$, the G-stable rank is additive*: *we have* $\mathrm{rk}^G_\alpha(v \boxplus w) = \mathrm{rk}^G_\alpha(v) + \mathrm{rk}^G_\alpha(w)$.

Proof. From Proposition 3.6 follows that $\mathrm{rk}^G_\alpha(v \boxplus w) \leq \mathrm{rk}^G_\alpha(v \boxplus 0) + \mathrm{rk}^G_\alpha(0 \boxplus w) \leq \mathrm{rk}^G_\alpha(v) + \mathrm{rk}^G_\alpha(w)$. Suppose that $g(t) \in G(K[\![t]\!])$ with $\mathrm{val}_t(g(t) \cdot (v \boxplus w)) = t^s$ for some $s > 0$. Assume that the block form of $g_i(t)$ with respect to the decomposition $V_i \oplus W_i$ is

$$g_i(t) = \begin{pmatrix} a_i(t) & b_i(t) \\ c_i(t) & d_i(t) \end{pmatrix}.$$

The $K[\![t]\!]$-module generated by the rows of $a_1(t)$ and $c_1(t)$ is a free submodule of $K[\![t]\!]^{n_1}$ of rank n_1, where $n_1 = \dim V_i$. Using the Smith normal form, there exist invertible matrices in $p(t) \in \mathrm{GL}_{n_1+m_1}(K[\![t]\!])$ and $q(t) \in \mathrm{GL}_{n_1}(K[\![t]\!])$ such that

$$\begin{pmatrix} a_1(t) \\ c_1(t) \end{pmatrix} = p(t) \begin{pmatrix} r(t) \\ 0 \end{pmatrix} q(t)$$

where $r(t)$ is an $n_1 \times n_1$ diagonal matrix. It follows that

$$p(t)^{-1} g_1(t) = \begin{pmatrix} r(t) & \star \\ 0 & \star \end{pmatrix}.$$

So without loss of generality, we may assume that $c_1(t) = 0$. A similar argument shows that we may assume without loss of generality that $b_2(t) = b_3(t) = \cdots = b_d(t) = 0$. If we project $g(t) \cdot v \boxplus w$ onto V, we get $a(t) \cdot v + b(t) \cdot w = a(t) \cdot v$ because $b_2(t) = 0$. This implies that $\mathrm{val}_t(a(t) \cdot v) \geq s$ and $\sum_{i=1}^d \alpha_i \,\mathrm{val}_t(\det a_i(t)) \geq s\,\mathrm{rk}^G_\alpha(v)$. Similarly, the projection of $g(t) \cdot v \boxplus w$ onto W is equal to $c(t) \cdot v + d(t) \cdot w = d(t) \cdot w$ because $c_1(t) = 0$. Therefore, we have $\mathrm{val}_t(d(t) \cdot w) \geq s$ and $\sum_{i=1}^d \alpha_i \,\mathrm{val}_t(\det d_i(t)) \geq s\,\mathrm{rk}^G_\alpha(w)$. Since $\det g_i(t) = \det a_i(t) \det d_i(t)$ because of the upper triangular or lower triangular form of $g_i(t)$, we get

$$\sum_{i=1}^s \alpha_i \,\mathrm{val}_t(\det g_i(t)) = \sum_{i=1}^s \alpha_i \,\mathrm{val}_t(\det a_i(t)) + \sum_{i=1}^s \alpha_i \,\mathrm{val}_t(\det d_i(t)) \geq s(\mathrm{rk}^G_\alpha(v) + \mathrm{rk}^G_\alpha(w)).$$

This proves that $\mathrm{rk}^G_\alpha(v \boxplus w) \geq \mathrm{rk}^G_\alpha(v) + \mathrm{rk}^G_\alpha(w)$. □

4. The stable T-rank

4A. ***The G-stable rank and the T-stable rank.*** The G-stable α-rank of a tensor v is the maximum of $\mu_\alpha(\lambda(t), v)$ where $\lambda(t)$ is a 1-parameter subgroup of G with $\mathrm{val}_t(\lambda(t) \cdot v) > 0$. A 1-parameter subgroup is contained in some maximal torus T (which itself is contained in some Borel subgroup B of G). We can fix a maximal torus T and consider all 1-parameter subgroups contained in T. Choosing a maximal torus of G corresponds to choosing a basis in each vector space V_i. So let us choose a basis in each V_i so that we can identify $\mathrm{GL}(V_i)$ with GL_{n_i}. Let $T_k \subseteq \mathrm{GL}_k$ be the subgroup of invertible diagonal $k \times k$ matrices, and $T = T_{n_1} \times T_{n_2} \times \cdots \times T_{n_d} \subseteq G$. Then T is a maximal torus of G.

Definition 4.1. We define the α-stable T-rank $\mathrm{rk}_\alpha^T(v)$ as the infimum over all $\mu_\alpha(\lambda(t), v)$ where $\lambda(t) \in T(K[t])$ is a 1-parameter subgroup of T with $\mathrm{val}_t(\lambda(t)\cdot v) > 0$.

Since every 1-parameter subgroup is conjugate to a 1-parameter subgroup in the maximal torus, we get the following corollary.

Corollary 4.2. *We have*

$$\mathrm{rk}_\alpha^G(v) = \inf_{g\in G} \mathrm{rk}_\alpha^T(g\cdot v).$$

4B. ***The T-stable rank and linear programming.*** For a tensor $v = (v_{i_1,i_2,\dots,i_d}) \in V = K^{n_1\times n_2\times\cdots\times n_d}$ we define its support by

$$\mathrm{supp}(v) = \{(i_1,\dots,i_d) \mid v_{i_1,i_2,\dots,i_d} \neq 0\}.$$

As we will see, $\mathrm{rk}_\alpha^T(v)$ only depends on $\mathrm{supp}(v)$ and α. For a nonnegative integer k, let $\underline{k} = \{1,2,\dots,k\}$. We will fix a support $S \subseteq \underline{n}_1 \times \underline{n}_2 \times\cdots\times \underline{n}_d$ and compute the corresponding α-stable T-rank.

Definition 4.3. Let $x(i,j)$ with $1 \le i \le d$ and $1 \le j \le n_i$ be real variables and $S \subseteq \underline{n}_1 \times\cdots\times \underline{n}_d$ be a support. The linear program $\mathrm{LP}_\alpha(S)$ asks to minimize $\sum_{i=1}^d \alpha_i \sum_{j=1}^{n_i} x(i,j)$ under the constraints:

(1) $x(i,j) \ge 0$ for $i = 1,2,\dots,d$ and $1 \le j \le n_i$.

(2) $\sum_{i=1}^d x(i,s_i) \ge 1$ for all $s \in S$.

Theorem 4.4. *If $v \in V$ has support S, then $\mathrm{rk}_\alpha^T(v)$ is the value of the linear program $\mathrm{LP}_\alpha(S)$.*

Proof. Suppose $\lambda(t) = (\lambda_1(t),\dots,\lambda_d(t)) \in T(K[t])$ is a 1-parameter subgroup, and $\lambda_i(t)$ is diagonal with entries $t^{x(i,1)}, t^{x(i,2)},\dots,t^{x(i,n_i)}$ where $x(i,j)$ is a nonnegative integer for all i,j. Also, assume that $\mathrm{val}_t(\lambda(t)\cdot v) = q > 0$ where v is a tensor with support S. This means that $\sum_{i=1}^d \alpha_i x(i,s_i) \ge q$ for all $(s_1,s_2,\dots,s_d) \in S$. We have $\mu_\alpha(\lambda(t),v) = \frac{1}{q}\bigl(\sum_{i=1}^d \alpha_i \sum_{j=1}^{n_i} x(i,j)\bigr)$ and $\mathrm{rk}_\alpha^T(v)$ is the infimum of all $\mu_\alpha(\lambda(t),v)$. If we replace $x(i,j)$ by $x(i,j)/q$, then we have $\sum_{i=1}^d \alpha_i x(i,s_i) \ge 1$ for all $(s_1,\dots,s_d) \in S$ and $\mu_\alpha(\lambda(t),v) = \sum_{i=1}^d \alpha_i \sum_{j=1}^{n_i} x(i,j)$. This shows that $\mathrm{rk}_\alpha^T(v)$ is the infimum of $\sum_{i=1}^d \alpha_i \sum_{j=1}^{n_i} x(i,j)$ under the constraints $x(i,j) \ge 0$ for all i,j, and $\sum_{i=1}^d x(i,s_i) \ge 1$ for all $s \in S$ for all i,j. This is the linear program $\mathrm{LP}_\alpha(S)$, except that the numbers $x(i,j)$ have to be rational. However, since the constraints are inequalities with coefficients in $\mathbb{Q}$, there exists an optimal solution over $\mathbb{Q}$. □

Example 4.5. Consider the tensor

$$v = [2,1,1] + [1,2,1] + [1,1,2] \in K^{2\times2\times2} = K^2 \otimes K^2 \otimes K^2.$$

with support $S = \{(2,1,1),(1,2,1),(1,1,2)\}$. We have to solve the following linear program $\mathrm{LP}(S) = \mathrm{LP}_{(1,1,1)}(S)$: minimize $\sum_{i=1}^3 \sum_{j=1}^2 x(i,j)$ under the constraints $x(i,j) \ge 0$ for $i = 1,2,3$ and $j = 1,2$ and

$$x(1,2) + x(2,1) + x(3,1) \ge 1$$
$$x(1,1) + x(2,2) + x(3,1) \ge 1$$
$$x(1,1) + x(2,1) + x(3,2) \ge 1$$

An optimal solution is $x(1,1)=x(2,1)=x(3,1)=\frac{1}{2}$ and $x(1,2)=x(2,2)=x(3,2)=0$. So the optimal value is $\mathrm{rk}^T(v)=3\cdot\frac{1}{2}=\frac{3}{2}$. It follows that $\mathrm{rk}^G(v)\le\mathrm{rk}^T(v)\le\frac{3}{2}$. It is easy to see that $\mathrm{srk}(v)>1$ (and thus equal 2). We will show that $\mathrm{rk}^G(v)=\frac{3}{2}$.

Suppose that $\mathrm{rk}^G(v)<\frac{3}{2}$. Then there exists a tensor $w\in K^{2\times2\times2}$ in the same G-orbit as v such that $\mathrm{rk}^T(w)<\frac{3}{2}$. Let $S'=\mathrm{supp}(w)\subseteq\underline{2}\times\underline{2}\times\underline{2}$ be the support of w. Also assume that $\{x(i,j)\}$ is an optimal solution for the linear program $\mathrm{LP}(S')$. By permuting coordinates, we may assume that $x(i,1)\ge x(i,2)$ for $i=1,2,3$. The support S' is not contained in $\{1\}\times\{1,2\}\times\{1,2\}$ because otherwise w and v would have slice rank 1. Therefore, $(2,i,j)\in S'$ for some i,j. Because of the ordering of the variables $x(i,j)$, $(2,1,1)\in S'$. Similarly, $(1,2,1),(1,1,2)\in S'$. Now $\mathrm{supp}(w)=S'\supseteq S=\mathrm{supp}(v)$, so $\mathrm{rk}^T(w)\ge\mathrm{rk}^T(v)=\frac{3}{2}$. Contradiction.

4C. *Comparison between the G-stable rank and the slice rank.* Besides the slice rank, we will also define a slice rank relative to a maximal torus T, or equivalently, relative to bases choices for $V_1,V_2,\dots,V_d$.

Definition 4.6. We say that a tensor v has T-slice rank 1 if v is contained in a space of the form

$$V_{i,j}=V_1\otimes V_2\otimes\cdots\otimes V_{i-1}\otimes[j]\otimes V_{i+1}\otimes\cdots\otimes V_d.$$

Now the T-slice rank $\mathrm{srk}^T(v)$ of an arbitrary tensor v is the smallest nonnegative integer r such that v is a sum of r tensors of T-slice rank 1.

The following result is clear from the definition of slice rank:

Corollary 4.7. *We have*

$$\mathrm{srk}(v)=\min_{g\in G}\mathrm{srk}^T(g\cdot v).$$

The T-slice rank of v depends only on its support $S=\mathrm{supp}(v)$ and can be expressed in terms of integer solutions of the linear program $\mathrm{LP}(S)$.

Proposition 4.8. *The T-slice rank $\mathrm{srk}^T(v)$ is the smallest possible value of $\sum_{i=1}^d\sum_{j=1}^{n_i}x(i,j)$ where the $x(i,j)$ satisfy the constraints*:

(1) $x(i,j)\in\{0,1\}$ *for $i=1,2,\dots,d$ and $1\le j\le n_i$.*

(2) $\sum_{i=1}^d x(i,s_i)\ge1$ *for all $s\in S$.*

Proof. Suppose that $x(i,j)\in\{0,1\}$ for all i,j. Define

$$V(x)=\sum_{\substack{i,j\\x(i,j)=1}}V_{i,j}.$$

A vector $[s_1,s_2,\dots,s_d]$ lies in $V(x)$ if and only if $\sum_{i=1}^d x(i,s_i)\ge1$. So a tensor v lies in $V(x)$ if and only if $\sum_{i=1}^d x(i,s_i)\ge1$ for all $s\in\mathrm{supp}(v)$. By definition, $\mathrm{srk}^T(v)$ is the smallest possible value of $\sum_{i,j}x(i,j)$ such that $v\in V(x)$. □

It is now easy to see that $\mathrm{rk}^T(v) \geq \frac{1}{d}\,\mathrm{srk}^T(v)$ (and this implies $\mathrm{rk}^G(v) \geq \frac{1}{d}\,\mathrm{srk}(v)$): If $x(i, j)$ is a solution to the linear program $\mathrm{LP}(S)$ where $S = \mathrm{supp}(v)$, then we define $x'(i, j) \in \{0, 1\}$ such that $x'(i, j) = 1$ if $x(i, j) \geq \frac{1}{d}$ and $x'(i, j) = 0$ otherwise. If $s \in S$ then we have $\sum_{i=1}^d x(i, s_i) \geq 1$. It follows that $x(i, s_i) \geq \frac{1}{d}$ for some i and $x'(i, s_i) = 1$ for some i. Therefore, $\sum_{i=1}^d x'(i, s_i) \geq 1$. Now $\mathrm{srk}^T(v) \leq \sum_{i,j} x'(i, j) \leq \sum_{i,j} d\,x(i, j) = d\,\mathrm{rk}^T(v)$. With a more refined argument, we can improve this bound.

Proposition 4.9. *For $d \geq 2$ we have $\mathrm{rk}^T(v) \geq \frac{2}{d}\,\mathrm{srk}^T(v)$ and therefore $\mathrm{rk}^G(v) \geq \frac{2}{d}\,\mathrm{srk}(v)$.*

Proof. Suppose that $x(i, j)$ is an optimal solution to the linear program. Note that $0 \leq x(i, j) \leq 1$ for all i, j. We define functions $f_1, f_2, \dots, f_d : [0, 1] \to \mathbb{R}$ by

$$f_i(\alpha) = |\{j \mid x(i, j) \geq \alpha\}|.$$

We have $\int_0^1 f_i(\alpha)\,d\alpha = \sum_j x(i, j)$. In particular, $\int_0^1 (f_1(\alpha) + \cdots + f_d(\alpha))\,d\alpha = \sum_{i,j} x(i, j)$. Let $s_i = 2i/(d(d-1))$ for $i = 0, 1, 2, \dots, d-1$. Note that $s_0 + s_1 + \cdots + s_{d-1} = 1$. We define a closed piecewise linear curve $\gamma = (\gamma_1, \dots, \gamma_d) : [0, d] \to \mathbb{R}^d$ with $\gamma(d) = \gamma(0) = [s_0, s_1, \dots, s_{d-1}]$, $\gamma(1) = [s_1, s_2, \dots, s_{d-1}, s_0]$, $\dots$, $\gamma(d-1) = [s_{d-1}, s_0, \dots, s_{d-2}]$ such that γ is linear on each of the intervals $[i, i+1]$, $i = 0, 1, \dots, d-1$. On the intervals $[0, 1], [1, 2], \dots, [d-1, d]$, $\gamma_i(t)$ goes through the intervals $[s_0, s_1]$,$[s_1, s_2]$,$\dots$, $[s_{d-2}, s_{d-1}]$,$[s_{d-1}, s_0]$ in some order. So $\frac{1}{d}\int_0^d f_i(\gamma_i(t))\,dt$ is the average of the averages of f_i of each of these d intervals. This is equal to the average value of $f_i(t)$ on the interval $[0, s_{d-1}] = \left[0, \frac{2}{d}\right]$:

$$\frac{1}{d}\int_0^d f_i(\gamma_i(t))\,dt = \frac{d}{2}\int_0^{2/d} f_i(t)\,dt \leq \frac{d}{2}\int_0^1 f_i(t)\,dt = \frac{d}{2}\sum_{j=1}^{n_i} x(i, j).$$

It follows that

$$\frac{1}{d}\int_0^d \left(\sum_{i=1}^d f_i(\gamma_i(t))\right) dt \leq \frac{d}{2}\sum_{i=1}^d \sum_{j=1}^{n_i} x(i, j) = \frac{d}{2}\,\mathrm{rk}^T(v).$$

Since the minimal value of $\sum_{i=1}^d f_i(\gamma_i(t))$ is at most the average, there exists a $t \in [0, d]$ such that $\sum_{i=1}^d f_i(\gamma_i(t)) \leq \frac{d}{2}\,\mathrm{rk}^T(v)$. Now define $x'(i, j) = 1$ if $x(i, j) \geq \gamma_i(t)$ and $x'(i, j) = 0$ if $x(i, j) < \gamma_i(t)$. If $s = (s_1, s_2, \dots, s_d) \in \mathrm{supp}(v)$, then $\sum_{i=1}^d x(i, s_i) \geq 1$. Since $\sum_{i=1}^d \gamma_i(t) = 1$, we have $x(i, s_i) \geq \gamma_i(t)$ for some i and $\sum_{i=1}^d x'(i, s_i) \geq 1$. We conclude that

$$\mathrm{srk}^T(v) \leq \sum_{i=1}^n \sum_{j=1}^{n_i} x'(i, j) = \sum_{i=1}^d f_i(\gamma_i(t)) \leq \frac{d}{2}\,\mathrm{rk}^T(v).$$

Finally, we get

$$\mathrm{srk}(v) = \inf_{g \in G} \mathrm{srk}^T(g \cdot v) \leq \frac{d}{2} \inf_{g \in G} \mathrm{rk}^T(g \cdot v) = \frac{d}{2}\,\mathrm{rk}^G(v). \qquad \square$$

4D. ***The dual program and the T-stable rank.***

Definition 4.10. For a support set S, the dual program $\mathrm{LP}^\vee_\alpha(S)$ is to maximize $\sum_{s\in S} y(s)$ under the constraints:

(1) $y(s)\geq 0$ for all $s\in S$.

(2) For all i,j we have
$$\sum_{\substack{s\in S\\ s_i=j}} y(s)\leq \alpha_i.$$

If x and y are optimal solutions for $\mathrm{LP}_\alpha(S)$ and $\mathrm{LP}^\vee_\alpha(S)$ respectively, then we have
$$\sum_{s\in S} y(s)=\sum_{i=1}^d \alpha_i\sum_{j=1}^{n_i} x(i,j)=\mathrm{rk}^T_\alpha(v)$$
and

(1) for all i,j, we have
$$\sum_{\substack{s\in S\\ s_i=j}} y(s)=\alpha_i \quad\text{or}\quad x(i,j)=0;$$

(2) for all $s\in S$ we have $\sum_{i=1}^d x(i,s_i)=1$ or $y(s)=0$.

4E. ***The supermultiplicative property of the T-stable rank.*** If $v\in V=V_1\otimes V_2\otimes\cdots\otimes V_d$ and $w\in W_1\otimes W_2\otimes\cdots\otimes W_d$ then we can consider the "vertical" tensor product $v\boxtimes w\in(V_1\otimes W_1)\otimes\cdots(V_d\otimes W_d)$.

Proposition 4.11. *We have* $\mathrm{rk}^T_{\alpha\beta}(v\boxtimes w)\geq \mathrm{rk}^T_\alpha(v)\,\mathrm{rk}^T_\beta(w)$, *where* $\alpha=(\alpha_1,\dots,\alpha_d)$, $\beta=(\beta_1,\dots,\beta_d)$ *and* $\alpha\beta=(\alpha_1\beta_1,\dots,\alpha_d\beta_d)$.

Proof. Let $S=\mathrm{supp}(v)$, $S'=\mathrm{supp}(w)$, $y(s)$, $s\in S$ be an optimal solution for the$\mathrm{LP}^\vee_\alpha(v)$ and $y'(s)$, $s\in S'$ be an optimal solution for $\mathrm{LP}^\vee_\beta(w)$. The tensor $v\boxtimes w$ has support $S\times S'$. For the dual program for $v\boxtimes w$ we have to maximize $\sum_{s\in S,s'\in S'} Y(s,s')$ under the constraints $Y(s,s')\geq 0$ for all $s\in S$, $s'\in S'$ and
$$\sum_{\substack{s\in S,s'\in S'\\ s_i=j,s'_i=j'}} Y(s,s')\leq \alpha_j\beta_{j'}$$
for all i,j,j'. One solution for this linear program is $Y(s,s')=y(s)y'(s')$. We get
$$\mathrm{rk}^T_{\alpha\beta}(v\boxtimes w)\geq\sum_{s\in S}\sum_{s'\in S'} Y(s,s')=\sum_{s\in S} y(s)\sum_{s'\in S'} y(s')=\mathrm{rk}^T_\alpha(v)\,\mathrm{rk}^T_\beta(w).\qquad\square$$

5. G-stable rank over $\mathbb{C}$

5A. ***Kempf–Ness theory.*** We recall some of the main results from Kempf–Ness theory [Kempf and Ness 1979; Woodward 2010]. Suppose that G is an complex reductive algebraic group with a maximal compact subgroup C and V is a representation of G. We fix a Hermitian inner product $\langle\cdot,\cdot\rangle$ on V that is invariant under C, i.e., $\langle g\cdot v, g\cdot w\rangle = \langle v, w\rangle$ for all $v, w \in V$ and $g \in C$. Let $\mathfrak{c}$ and $\mathfrak{g}$ be the Lie algebras of C and G respectively, and let $\mathfrak{c}^\star$ be the dual space of $\mathfrak{c}$. We have $\mathfrak{g} = \mathfrak{c}\oplus i\mathfrak{c}$. For $v \in V$, we define a morphism $\psi_v : G \to \mathbb{R}$ by $g \mapsto \|g\cdot v\|^2 = \langle g\cdot v, g\cdot v\rangle$. The differential $(d\psi_v)_I : \mathfrak{g} \to \mathbb{R}$ of ψ_v at the identity $I \in G$ is given by

$$(d\psi_v)_I : \xi \mapsto \langle \xi v, v\rangle + \langle v, \xi v\rangle \in \mathbb{R}.$$

Because $\|g\cdot v\|^2$ is constant on C, $(d\psi_v)_I$ vanishes on $\mathfrak{c}$. So $\langle v, \xi v\rangle = -\langle \xi v, v\rangle$ for $\xi \in \mathfrak{c}$. If $\xi \in \mathfrak{c}$ then we have $(d\psi_v)_I(i\xi) = \langle i\xi v, v\rangle + \langle v, i\xi v\rangle = i\langle \xi v, v\rangle - i\langle v, \xi v\rangle = 2i\langle \xi v, v\rangle$. For the following result, see [Woodward 2010, Corollary 5.2.5].

Theorem 5.1 (Kempf and Ness). *An orbit $G\cdot v$ is closed if and only there exists $w \in G\cdot v$ with $(d\psi_w)_I = 0$.*

Let $V = V_1\otimes V_2\otimes\cdots\otimes V_d$ with $V_i = \mathbb{C}^{n_i}$. For $v \in V$, let $\Phi_i(v) \in (V_1\otimes\cdots\otimes \widehat{V_i}\otimes\cdots\otimes V_d)^\star \to V_i$ be the i-th flattening of v.

5B. *A formula for the G-stable rank over* $\mathbb{C}$. We will use Kempf–Ness theory to prove the following theorem:

Theorem 5.2. *For $\alpha \in \mathbb{R}_{>0}$ we have*

$$\operatorname{rk}^G_\alpha(v) = \sup_{g\in G}\min_i \frac{\alpha_i\|g\cdot v\|^2}{\|\Phi_i(g\cdot v)\|^2_\sigma}.$$

For the proof of the theorem, we need the following lemma:

Lemma 5.3. *Suppose that $\beta \in \mathbb{Q}^d_{>0}$, $r = \frac{p}{q}$ with p, q positive integers, $q\beta \in \mathbb{Z}^d$ and $v \in V = V_1\otimes V_2\otimes\cdots\otimes V_d$. As in Proposition 2.3, let*

$$W = (V^{\otimes p}\otimes \det{}^{-q\beta})\oplus V_1^{n_1}\oplus V_2^{n_2}\oplus\cdots\oplus V_d^{n_d}.$$

and $w = (v^{\otimes p}\otimes 1, u_1, \dots, u_d)$. Define $\psi_w : G \to W$ by $\psi_w(g) = g\cdot w$. Then we have $(d\psi_w)_I = 0$ if and only if

$$p\|v\|^{2p-2}\Phi_i(v)\Phi_i^\star(v) - q\beta_i\|v\|^{2p}I_{n_i} + u_iu_i^\star = 0$$

for all i.

Proof. The Hermitian scalar products on $V_1, V_2, \dots, V_d$ induce Hermitian scalar products on $V_1^{n_1}, \dots, V_d^{n_d}$, V, $V^{\otimes p}$, $V^{\otimes p}\otimes\det^{-q\beta}$ and W in a natural way. We have

$$\|w\|^2 = \|v\|^{2p} + \sum_{i=1}^d \|u_i\|^2$$

and

$$\psi_w(g) = \|g\cdot w\|^2 = \|g\cdot v\|^{2p}\det{}^{-2q\beta}(g) + \sum_{i=1}^d \|g_iu_i\|^2.$$

The Lie algebra of G can be identified with

$$\mathfrak{g} = \mathrm{End}(V_1) \oplus \mathrm{End}(V_2) \oplus \cdots \oplus \mathrm{End}(V_d).$$

The Lie algebra $\mathfrak{c}$ consists of all d-tuples $(\xi_1, \ldots, \xi_d)$ of skew-Hermitian matrices, and $i\mathfrak{c}$ consists of d-tuples of Hermitian matrices. We compute the differential $(d\psi_w)_I$. Note that $\mathrm{GL}(V_i)$ acts on the i-th mode. If we view v as the flattened tensor $\Phi_i(v)$, then g_i acts just by left multiplication: $\Phi_i(g_i \cdot v) = g_i \Phi_i(v)$. Let $\mathrm{Tr}(\cdot)$ denote the trace. The differential of $g_i \mapsto \|g_i \cdot v\|^2 = \mathrm{Tr}(g_i \Phi_i(v) \Phi_i^\star(v) g_i^\star)$ at the identity is given by $\xi_i \in \mathrm{End}(V_i) \mapsto \mathrm{Tr}(\xi_i \Phi_i(v) \Phi_i^\star(v)) + \mathrm{Tr}(\Phi_i(v)\Phi_i^\star(v)\xi_i^\star)$. If we restrict to Hermitian ξ_i, then this is equal to $2\,\mathrm{Tr}(\xi_i \Phi_i(v)\Phi_i^\star(v))$. The differential of $\|g \cdot v\|^2$ restricted to $i\mathfrak{c} \subseteq \mathfrak{g}$ is $(\xi_1, \ldots, \xi_d) \mapsto 2\sum_{i=1}^d \mathrm{Tr}(\xi_i \Phi_i(v)\Phi_i^\star(v))$. The differential of $g_i \mapsto \det(g_i)$ at the identity is $\xi_i \mapsto \mathrm{Tr}(\xi_i)$. Combining these results with the product rule of differentiation, we get for $\xi \in i\mathfrak{c}$ that

$$\begin{aligned}(d\phi_w)_I(\xi) &= \sum_{i=1}^d (2p\|v\|^{2p-2}\,\mathrm{Tr}(\xi_i \Phi_i(v)\Phi_i^\star(v)) - 2q\beta_i q\|v\|^{2p}\,\mathrm{Tr}(\xi_i) + 2\,\mathrm{Tr}(\xi_i u_i u_i^\star)) \\ &= \sum_{i=1}^d \langle \xi_i, \|v\|^{2p-2}\Phi_i(v)\Phi_i^\star(v) - 2q\beta_i\|v\|^{2p} I_{n_i} + 2u_i u_i^\star\rangle.\end{aligned}$$

We have $(d\phi_w)_I = 0$ if and only if

$$2p\|v\|^{2p-2}\Phi_i(v)\Phi_i^\star(v) - 2q\beta_i\|v\|^{2p} I_{n_i} + 2u_i u_i^\star = 0$$

for all i. □

Proof of Theorem 5.2. Let us define

$$f_\alpha(v) = \sup_{g \in G} \min_i \frac{\alpha_i \|g \cdot v\|^2}{\|\Phi_i(g \cdot v)\|_\sigma^2}$$

Suppose that $r \in \mathbb{Q}$ and $f_\alpha(v) \le r$. Assume that $\beta \in \mathbb{Q}_{>0}^d$ with $\beta_i > \alpha_i$ for all i. We can write $r = p/q$ such that $p, q \in \mathbb{Z}$ are positive and $q\beta_i \in \mathbb{Z}$ for all i. From $f_\alpha(v) \le r$ follows that

$$\alpha_i \|g \cdot v\|^2 I_{n_i} - r\Phi_i(g \cdot v)\Phi_i^\star(g \cdot v)$$

is nonnegative definite for all i. This implies that

$$\beta_i \|g \cdot v\|^2 I_{n_i} - r\Phi_i(g \cdot v)\Phi_i^\star(g \cdot v)$$

is positive definite for all i. Multiplying with $p\|g \cdot v\|^{2p-2}$ we get that

$$p\beta_i \|g \cdot v\|^{2p} I_{n_i} - q\|g \cdot v\|^{2p-2}\Phi_i(g \cdot v)\Phi_i^\star(g \cdot v)$$

is positive definite and equal to $u_i u_i^\star$ for some $u_i \in V_i^{n_i}$. This shows that $(d\psi_{g \cdot w})_I = 0$. By Theorem 5.1, the G-orbit of w is closed. By Proposition 2.3, we have $\mathrm{rk}_\beta^G(v) \ge r$. Because this is true for every rational $\beta > \alpha$, we get $\mathrm{rk}_\alpha^G(v) \ge r$. Since this is true for any $r \in \mathbb{Q}$ with $r \ge f_\alpha(v)$, we can conclude that $\mathrm{rk}_\alpha^G(v) \ge f_\alpha(v)$.

Suppose that $\beta \in \mathbb{Q}^d_{>0}$ and $\beta_i < \alpha_i$ for all i. Let $r = \mathrm{rk}^G_\beta(v) < \mathrm{rk}^G_\alpha(v)$. We can write $r = \frac{p}{q}$ such that p, q are positive integers, and $q\beta \in \mathbb{Z}^d$. We can choose an invertible $u_i \in V_i^{n_i}$ for all i. Now

$$w = (v^{\otimes p} \otimes 1, u_1, u_2, \dots, u_d) \in (V^{\otimes p} \otimes \det^{-q\beta}) \oplus V_1^{n_1} \oplus V_2^{n_2} \oplus \cdots \oplus V_d^{n_d}$$

is G-semistable by Proposition 2.3. So there exists a nonzero $w' \in \overline{G \cdot w}$ with $(d\psi_{w'})_I = 0$. We can write $w' = ((v')^{\otimes d}, u'_1, \dots, u'_d)$. Using Lemma 5.3, we get

$$p\|v'\|^{2p-2}\Phi_i(v')\Phi_i^\star(v') - q\beta_i\|v'\|^{2p} I_{n_i} + u'_i(u'_i)^\star = 0.$$

So

$$q\beta_i\|v'\|^{2p} I_{n_i} - p\|v'\|^{2p-2}\Phi_i(v')\Phi_i^\star(v')$$

is nonnegative definite for all i. Therefore,

$$q\alpha_i\|v'\|^{2p} I_{n_i} - p\|v'\|^{2p-2}\Phi_i(v')\Phi_i^\star(v')$$

is positive definite for all i.

Since w' lies in $\overline{G \cdot w}$, there exists a $g \in G$ such that

$$q\alpha_i\|g \cdot v\|^{2p} I_{n_i} - p\|g \cdot v\|^{2p-2}\Phi_i(g \cdot v)\Phi_i^\star(g \cdot v)$$

is positive definite for all i. It follows that

$$\|\Phi_i(g \cdot v)\|_\sigma^2 = \|\Phi_i(g \cdot v)\Phi_i^\star(g \cdot v)\|_\sigma \le \frac{q\alpha_i\|g \cdot v\|^{2p}}{p\|g \cdot v\|^{2p-2}} = \frac{\alpha_i\|g \cdot v\|^2}{r}$$

for all i and

$$\min_i \frac{\alpha_i\|g \cdot v\|^2}{\|\Phi_i(g \cdot v)\|_\sigma^2} \ge r.$$

This shows that $f_\alpha(v) \ge r = \mathrm{rk}^G_\beta(v)$. Since $\beta \in \mathbb{Q}^d_{>0}$ was arbitrary with $\beta < \alpha$, we obtain $f_\alpha(v) \ge \mathrm{rk}^G_\alpha(v)$. We conclude that $f_\alpha(v) = \mathrm{rk}^G_\alpha(v)$. □

5C. ***The supermultiplicative property of the G-stable rank in characteristic 0.***

Theorem 5.4. *If $v \in V_1 \otimes V_2 \otimes \cdots \otimes V_d$ and $w \in W_1 \otimes W_2 \otimes \cdots \otimes W_d$ where $V_1, \dots, V_d, W_1, \dots, W_d$ are $\mathbb{C}$-vector spaces and $\alpha, \beta \in \mathbb{R}^d_{>0}$, then we have*

$$\mathrm{rk}^G_{\alpha\beta}(v \boxtimes w) \ge \mathrm{rk}^G_\alpha(v)\,\mathrm{rk}^G_\beta(w).$$

Proof. if $g \in \mathrm{GL}(V_1) \times \cdots \times \mathrm{GL}(V_d)$ and $h \in \mathrm{GL}(W_1) \times \cdots \times \mathrm{GL}(W_d)$ then we can consider $g \boxtimes h \in \mathrm{GL}(V_1 \otimes W_1) \times \cdots \times \mathrm{GL}(V_d \otimes W_d)$. We have

$$\frac{\alpha_i\beta_i\|(g \boxtimes h)\cdot(v \boxtimes w)\|^2}{\|\Phi_i((g \boxtimes h)\cdot(v \boxtimes w))\|_\sigma} = \frac{\alpha_i\beta_i\|((g \cdot v) \boxtimes (h \cdot w)\|^2}{\|\Phi_i((g \cdot v) \boxtimes (h \cdot w))\|_\sigma} = \frac{\alpha_i\|g \cdot v\|^2\beta_i\|h \cdot w\|^2}{\|\Phi_i(g \cdot v)\|_\sigma\|\Phi_i(h \cdot w)\|_\sigma}.$$

Therefore, we get

$$\min_i \frac{\alpha_i\beta_i\|(g\boxtimes h)\cdot(v\boxtimes w)\|^2}{\|\Phi_i((g\boxtimes h)\cdot(v\boxtimes w))\|_\sigma} \geq \min_i \frac{\alpha_i\|g\cdot v\|^2}{\|\Phi_i(g\cdot v)\|_\sigma}\cdot \min_j \frac{\beta_j\|h\cdot w\|^2}{\|\Phi_j(h\cdot w)\|_\sigma}.$$

Taking the supremum over all g and h now gives $\mathrm{rk}^G_{\alpha\beta}(v\boxtimes w)\geq \mathrm{rk}^G_\alpha(v)\,\mathrm{rk}^G_\beta(w)$. □

6. Application of the G-stable rank to the cap set problem

The cap set problem asks for a largest possible subset $S\subseteq \mathbb{F}_3^n$ without an arithmetic progression. Let $c(n)$ be the largest possible cardinality of such a set. It was recently proved by Ellenberg and Gijswijt that $c(n)=O(\theta^n)$, where $\theta=\frac{3}{8}(207+33\sqrt{33})^{1/3}<2.756$. Tao gave an elegant formulation of the proof of this bound using the notion of slice rank. Here we will use a similar approach, using the G-stable rank instead of the slice rank to get an explicit bound for all n which the same asymptotic behavior. We view K^3 as the vector space with basis [0], [1], [2] where we view 0, 1, 2 as elements in $\mathbb{F}_3$. More generally, we view K^{3^n} as the vector space with basis $[a]$, $a\in\mathbb{F}_3^n$. Note that a, b, c form an arithmetic progression in $\mathbb{F}_3^n$ if and only if $a+b+c=0$. Consider the tensor

$$v_n=\sum_{\substack{(a,b,c)\in\mathbb{F}_3^{n\times 3}\\ a+b+c=0}}[a]\otimes[b]\otimes[c]=\sum_{\substack{(a,b,c)\in\mathbb{F}_3^{n\times 3}\\ a+b+c=0}}[a,b,c]\in K^{3^n}\otimes K^{3^n}\otimes K^{3^n}.$$

Suppose that $S\subset\mathbb{F}_3^n$ is a set without arithmetic progression. Then we have

$$w=\sum_{\substack{(a,b,c)\in S^3\\ a+b+c=0}}[a,b,c]\in K^3\otimes K^3\otimes K^3=\sum_{a\in S}[a,a,a].$$

The tensor w is a projection of v and lies in the orbit closure of v. In particular, we have $\mathrm{rk}^G(w)\leq\mathrm{rk}^G(v)$. Since w is a direct sum of $|S|$ rank 1 tensors, we get $\mathrm{rk}^G(w)\geq|S|$ by Proposition 3.8. So we have $\mathrm{rk}^G(v)\geq\mathrm{rk}^G(w)\geq|S|$.

We will work over the field $K=\mathbb{F}_3$. For a function $f:\mathbb{F}_3^n\to\mathbb{F}_3$ we define

$$\langle f\rangle=\sum_{a\in\mathbb{F}_3^n}f(a)[a]\in K^{3^n}.$$

In particular, we have $\langle 1\rangle=[0]+[1]+[2]$, $\langle x\rangle=[1]+2[2]=[1]-[2]$ and $\langle x^2\rangle=[1]+[2]$. A basis of K^{3^n} is formed by taking all $\langle p(x)\rangle$ where $p(x)=p(x_1,\ldots,x_n)$ is a polynomial of degree ≤ 2 in each of the variables $x_1,x_2,\ldots,x_n$. With respect to the basis $\langle 1\rangle,\langle x\rangle,\langle x^2\rangle$, we have $v_n=\langle f\rangle$ where $f:\mathbb{F}_3^n\times\mathbb{F}_3^n\times\mathbb{F}_3^n\to\mathbb{F}_3$ is given by

$$f(x,y,z)=\begin{cases}1 & \text{if } x+y+z=0;\\ 0 & \text{otherwise.}\end{cases}$$

For $n=1$ we have $v_1=\langle f\rangle$ where $f:\mathbb{F}_3\times\mathbb{F}_3\times\mathbb{F}_3\to\mathbb{F}_3$ is given by $f(x,y,z)=1-(x+y+z)^2=1-x^2-y^2-z^2+x+y+z$. So we have

$$v_1=\langle 1,1,1\rangle-\langle x^2,1,1\rangle-\langle 1,x^2,1\rangle-\langle 1,1,x^2\rangle+\langle 1,x,x\rangle+\langle x,1,x\rangle+\langle x,x,1\rangle.$$

The support of S with respect to the basis $\langle 1\rangle$, $\langle x\rangle$, $\langle x^2\rangle$ is

$$\{(0,0,0),(2,0,0),(0,2,0),(0,0,2),(0,1,1),(1,0,1),(1,1,0)\}.$$

An optimal solution to the linear program is $x(1,0)=x(2,0)=x(3,0)=\frac{1}{2}$, $x(1,1)=x(2,1)=x(3,1)=\frac{1}{4}$ and $x(1,2)=x(2,2)=x(3,2)=0$, which gives $\mathrm{rk}^G(v)\geq\mathrm{rk}^T(v)=\sum_{i,j}x(i,j)=\frac{9}{4}=2.25$. An optimal solution for the dual program is $y(2,0,0)=y(0,2,0)=y(0,0,2)=\frac{1}{4}$ and $y(0,1,1)=y(1,0,1)=y(1,1,0)=\frac{1}{2}$ and $y(0,0,0)=0$.

The support of the tensor $v^{\boxtimes n}=v\boxtimes v\boxtimes\cdots\boxtimes v$ is contained in the set

$$T_n=\{(\lambda,\mu,\nu)\in(\{0,1,2\}^n)^3\mid|\lambda|\leq 2n,|\mu|\leq 2n,|\nu|\leq 2n\}.$$

We will give a solution to the linear program $\mathrm{LP}(S^n)$ that we conjecture to be optimal. Whether optimal or not, it will give an upper bound for the G-stable rank of $v^{\boxtimes n}$. Suppose that $t_0,t_1,t_2,\ldots,t_{2n}\geq 0$ are numbers such that $t_i+t_j+t_k\geq 1$ whenever $i+j+k\leq 2n$. If we define $x(i,\lambda)=t_{|\lambda|}$ for all $\lambda\in\{0,1,2\}^n$, and $i=1,2,3$ then we have $x(1,\lambda)+x(2,\mu)+x(3,\nu)=t_{|\lambda|}+t_{|\mu|}+t_{|\nu|}\geq 1$, so we have a solution to the linear program. So we get

$$\mathrm{rk}^G(v)\leq\sum_{i=1}^{3}\sum_{\lambda}x(i,\lambda)=3\sum_{\lambda}t_{|\lambda|}=3\sum_{i=0}^{2n}f_{n,i}t_i$$

where $f_{n,i}$ is the number of solutions to $a_1+a_2+\cdots+a_n=d$ with $a_1,a_2,\ldots,a_n\in\{0,1,2\}$. So $f_{n,i}$ is the coefficient of x^i in $(1+x+x^2)^n$. To choose the t optimally, we have to solve a linear program by minimizing $3\sum_{i=0}^{2n}f_{n,i}t_i$ under the constraints:

(1) $t_i+t_j+t_k\geq 1$ if $i+j+k\leq 2n$.

(2) $t_i\geq 0$ for all i.

The optimal solutions for the t_i are given in Table 2.

In Table 2, the column UB gives the value of $3\sum_{i=0}^{2n}f_{n,i}t_i$ which is an upper bound for the G-stable rank and the cardinality of a cap set in $\mathbb{F}_3^n$. The column labeled "best cap set" gives the cardinality of the largest known cap set in $\mathbb{F}_3^n$. The column EG gives the Ellenberg–Gijswijt upper bound, which is $3\sum_{i=0}^{\lfloor(2/3)n\rfloor}f_{n,i}$. This estimate relies on the fact that if i,j,k are nonnegative integers with $i+j+k\leq 2n$, then it follows that $\min\{i,j,k\}\leq\left\lfloor\frac{2n}{3}\right\rfloor$. But one can say something stronger, namely $i\leq\left\lfloor\frac{2n}{3}\right\rfloor$, $j\leq\left\lfloor\frac{2n-1}{3}\right\rfloor$ or $k\leq\left\lfloor\frac{2n-2}{3}\right\rfloor$. This observation gives a better bound that is still based on the slice rank in the column labeled EG'. In the comment section of [Tao 2016], Fedor Petrov gives a refined argument to improve on that of Ellenberg and Gijswijt to $2\sum_{i=0}^{\lfloor(2/3)n\rfloor}f_{n,i}$, an improvement by a factor $\frac{2}{3}$. This bound is given in the column labeled P. In fact, the discussion of Petrov and Tao shows that we get an even better upper bound if we minimize $\sum_{i=0}^{m}f_{n,i}+\sum_{i=0}^{2n-2-2m}f_{n,i}$ over all m with $0\leq m<n$. This bound is given in the column P'.

n		0	1	2	3	4	5	6	UB	P'	P	EG'	EG	best cap set
1	$f_{1,i}$	1	1	1	0	0	0	0	$2\frac{1}{4}$	2	2	3	3	2
	t_i	$\frac{1}{2}$	$\frac{1}{4}$	0	0	0	0	0						
2	$f_{2,i}$	1	2	3	2	1	0	0	6	4	6	7	9	4
	t_i	$\frac{3}{5}$	$\frac{2}{5}$	$\frac{1}{5}$	0	0	0	0						
3	$f_{3,i}$	1	3	6	7	6	3	1	15	11	20	18	30	9
	t_i	1	$\frac{2}{3}$	$\frac{1}{3}$	0	0	0	0						
4	$f_{4,i}$	1	4	10	16	19	16	10	39	30	30	45	45	20
	t_i	1	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	0	0	0						
5	$f_{5,i}$	1	5	15	30	45	51	45	105	72	102	123	153	45
	t_i	1	$\frac{4}{5}$	$\frac{3}{5}$	$\frac{2}{5}$	$\frac{1}{5}$	0	0						
6	$f_{6,i}$	1	6	21	50	90	126	141	274	196	336	324	504	112
	t_i	1	1	1	$\frac{2}{3}$	$\frac{1}{3}$	0	0						

Table 2. Optimal solutions for the t_i.

In the table of Section 1E we have computed the optimal value of $3\sum_{i=0}^{2n} f_{n,i}t_i$ rounded down to the nearest integer for $n \le 20$. This bound is an upper bound for the cardinality of a cap set in $\mathbb{F}_3^n$.

Looking at optimal solutions for small n, we make the following conjecture:

Conjecture 6.1. *The optimal solution of the linear program for $t_0, t_1, t_2, \ldots, t_{2n}$ is as follows*:

$$p\begin{cases} \underbrace{1, 1, \ldots, 1}_{(2n-3)/3}, \frac{2}{3}, \frac{1}{3}, 0, 0, \ldots & \textit{if } n \equiv 0 \bmod 3, \\ \underbrace{1, 1, \ldots, 1}_{(2n-5)/3}, \frac{3}{4}, \frac{1}{2}, \frac{1}{4}, 0, 0, \ldots & \textit{if } n \equiv 1 \bmod 3, \\ \underbrace{1, 1, \ldots, 1}_{(2n-7)/3}, \frac{4}{5}, \frac{3}{5}, \frac{2}{5}, \frac{1}{5}, 0, 0, \ldots & \textit{if } n \equiv 2 \bmod 3. \end{cases}$$

7. Conclusion and further directions

The G-stable rank is a new notion of rank for tensors. Up to a constant it is equal to the slice rank, but it is more refined in the sense that it can take noninteger values, and unlike the slice rank it is supermultiplicative with respect to vertical tensor products. As an illustration, we showed that the G-stable rank can be used to improve upper bounds for the cardinality of cap sets. Zhi Jiang recently proved Conjecture 6.1 in [Jiang 2021]. He also improved the asymptotic upper bound of Ellenberg and Gijswijt to suggest an upper bound of the form $C\theta^n/\sqrt{n}$ where C is some explicit constant. Since the asymptotic subrank of the cap set tensor is θ, the approach with G-stable rank cannot give an upper bound $O(\gamma^n)$ where $\gamma < \theta$.

Besides algebraic applications of tensor decompositions there are also many numerical applications such as psychometrics [Tucker 1963; 1964; 1966; Carroll and Chang 1970; Harshman 1970] and chemometrics [Appellof and Davidson 1981]. For more details and references, see the survey article [Kolda and Bader 2009] or the books [Kroonenberg 2008; Landsberg 2012]. Formula (2) allows us to compute or approximate the G-stable rank for real or complex tensors using optimization. Future directions of research include algorithms for approximating the G-stable rank of a tensor, or to approximate a given tensors by tensors of low G-stable rank and apply these to such tasks as denoising, dimension reduction and tensor completion.

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