

Global holomorphic linearization of actions of compact Lie groups on \mathbb{C}^n

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1. INTRODUCTION

The group $\text{Aut}_{\text{hol}}(\mathbb{C}^n)$ of holomorphic automorphisms of \mathbb{C}^n is very large and complicated for $n \geq 2$. One attempt to study it, is to study its finite subgroups or more general its subgroups which are isomorphic to compact Lie groups. The compact subgroups of $\text{GL}_n(\mathbb{C}) < \text{Aut}_{\text{hol}}(\mathbb{C}^n)$ are much easier to understand. If one has a compact subgroup $K < \text{GL}_n(\mathbb{C})$, then for each $\alpha \in \text{Aut}_{\text{hol}}(\mathbb{C}^n)$ the conjugated subgroup $\alpha K \alpha^{-1}$ provides another compact subgroup of $\text{Aut}_{\text{hol}}(\mathbb{C}^n)$ (isomorphic as a Lie group to K). So it is natural to ask whether all subgroups of $\text{Aut}_{\text{hol}}(\mathbb{C}^n)$ isomorphic to compact Lie groups arise in this way:

Holomorphic Linearization Problem. *Let $K \hookrightarrow \text{Aut}_{\text{hol}}(\mathbb{C}^n)$ be a subgroup of the group $\text{Aut}_{\text{hol}}(\mathbb{C}^n)$ of holomorphic automorphisms of \mathbb{C}^n , which is isomorphic to a compact Lie group. Can one conjugate this subgroup K by a single automorphism into the general linear group $\text{GL}_n(\mathbb{C}) < \text{Aut}_{\text{hol}}(\mathbb{C}^n)$?*

This question can be formulated in another equivalent way. For that we introduce the following notation.

We say that a real Lie group G acts on a complex space X by holomorphic transformations if the action map $\phi : G \times X \rightarrow X$ (which by definition satisfies $\phi(g, \phi(h, x)) = \phi(gh, x)$, $\phi(e, x) = x \forall x \in X, g, h \in G$) is real analytic and for all $g \in G$ the map $\phi(g, \cdot) : X \rightarrow X$ is holomorphic (in fact a holomorphic automorphism).

REMARK 1.1. Every continuous action of a real Lie group on a complex space by holomorphic automorphisms is already real analytic ([2] 1.6.). So it would be the same as demanding that the action map $\phi : G \times X \rightarrow X$ is only continuous.

If X is a complex manifold the compact-open topology makes the group $\text{Aut}_{\text{hol}}(X)$ of holomorphic automorphisms of X a topological group. The action map $\phi : G \times X \rightarrow X$ is continuous iff the corresponding group homomorphism of G into $\text{Aut}_{\text{hol}}(X)$ is continuous. By the above remark we see that an action of a real Lie group G on \mathbb{C}^n by holomorphic transformations is the same as a continuous group homomorphism $\alpha : G \rightarrow \text{Aut}_{\text{hol}}(\mathbb{C}^n)$. An effective action of a Lie group G on \mathbb{C}^n is the same as a subgroup of $\text{Aut}_{\text{hol}}(\mathbb{C}^n)$ which is as a topological group (topology induced from that of $\text{Aut}_{\text{hol}}(\mathbb{C}^n)$) isomorphic to G .

So the Holomorphic Linearization Problem asks whether each action of a compact Lie group K on \mathbb{C}^n is linear after some holomorphic change of coordinates on \mathbb{C}^n , i.e. is linearizable.

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Under special suppositions (fixing a dimension n , K is already contained in some proper subgroup of $\text{Aut}_{\text{hol}}(\mathbb{C}^n)$, etc.) the answer to this question can be yes. We sketch the main positive results on linearization in the second section. Also for the convenience of the reader we recall there some facts about holomorphic actions of compact Lie groups on Stein manifolds.

It was shown recently by the authors that unfortunately the general answer to the Holomorphic Linearization Problem is no. Even if one specifies the isomorphism type of K (as a Lie group), the general answer is no for each single type. So it turns out that one has to specify both the dimension n and the type of K to obtain a positive answer. More precise we proved in [5]:

Theorem 1.2. *For every compact Lie group K (except the trivial group) there exists a natural number N_K such that for all $n \geq N_K$ there exists an effective non-linearizable action of K on \mathbb{C}^n by holomorphic automorphisms.*

In fact the result in [5] is formulated for complex reductive groups but the non-linearizable actions of compact Lie groups are in one to one correspondence to non-linearizable actions of complex reductive Lie groups (see [17] for a discussion of this correspondence).

One aim of this article is to explain very explicitly the (up to now only known) method of construction of non-linearizable K -actions on \mathbb{C}^n in the case of the simplest positive dimensional compact Lie group $K \cong S^1 := \{z \in \mathbb{C} : |z| = 1\}$ (see section 4). Moreover we go a step further:

Suppose we fix such a dimension $n_0 \geq 2$ and such a type of compact Lie groups that there are compact subgroups of $\text{Aut}_{\text{hol}}(\mathbb{C}^{n_0})$ of that type which are not conjugate in $\text{Aut}_{\text{hol}}(\mathbb{C}^{n_0})$ to a subgroup of $\text{GL}_{n_0}(\mathbb{C})$. It is natural to ask:

How many conjugacy classes form the subgroups of that fixed type in the holomorphic automorphism group of \mathbb{C}^{n_0} ?

Observe that there are only countably many conjugacy classes of compact subgroups of $\text{GL}_{n_0}(\mathbb{C})$ of a fixed type. For example every linear representation of S^1 on \mathbb{C}^n is (since S^1 is abelian) conjugate to a diagonal one, i.e., to some representation

$$\lambda \mapsto \begin{pmatrix} \lambda^{\alpha_1} & & \\ & \ddots & \\ & & \lambda^{\alpha_n} \end{pmatrix}, \quad \lambda \in S^1 \quad \text{for some exponents } (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n.$$

That means the conjugacy classes of S^1 in $\text{GL}_n(\mathbb{C})$ are parameterized by n -tuples of integers which have no common divisor (to make the homomorphism injective). We will show: *There are uncountably many conjugacy classes of S^1 in $\text{Aut}_{\text{hol}}(\mathbb{C}^n)$ for all $n \geq 4$ (Corollary 4.6).*

A crucial point for the construction of non-linearizable K -actions on \mathbb{C}^n is the existence of proper holomorphic embeddings $\phi : \mathbb{C} \hookrightarrow \mathbb{C}^k$ which are not equivalent to the standard embedding $\mathbb{C} \hookrightarrow \mathbb{C}^k$, $\xi \mapsto (\xi, 0, \dots, 0)$ proved by FORSTNERIC, GLOBEVNIK and ROSAY in [7] (see also [19]). We will use their technique (referring only to the harder technical results from [18] and [4]) to prove the existence of uncountably many non-equivalent embeddings of \mathbb{C} into \mathbb{C}^n for $n \geq 2$ (Proposition 3.5). This we do in section 3. In section 4 we explain how to get non-conjugate S^1 -actions from non-equivalent embeddings.

2. POSITIVE RESULTS ON HOLOMORPHIC LINEARIZATION

We recall the notion of categorical quotient for an action $G \times X \rightarrow X$ of a Lie group G on a complex space X by holomorphic transformations.

Definition. A complex space $X//G$ together with a G -invariant holomorphic map $\pi_X : X \rightarrow X//G$ is called categorical quotient for the action $G \times X \rightarrow X$ if it satisfies the following universal property:

For every holomorphic G -invariant map $\psi : X \rightarrow Y$ from X to some complex G -space Y there exists a unique holomorphic map $\tilde{\psi} : X//G \rightarrow Y$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\psi} & Y \\ \pi_X \searrow & & \nearrow \tilde{\psi} \\ & X//G & \end{array}$$

commutes.

The existence of the categorical quotient in the case where G is a complex reductive group acting holomorphically on a Stein space X was proved by SNOW [21] and if G is a compact group acting by holomorphic transformations on a Stein space X by HEINZNER [8]. As a topological space $X//G$ is just the topological quotient of X with respect to the equivalence relation R associated to the algebra $\mathcal{O}^G(X)$ of G -invariant holomorphic functions on X :

$$R = \{(x, y) \in X \times X, f(x) = f(y) \quad \forall f \in \mathcal{O}^G(X)\}$$

Now we recall some facts about the categorical quotient for an action of a compact Lie group K on a Stein space X in the special situation where the K -action extends to an action of the universal complexification group $K^{\mathbb{C}}$ (for definition see [10], in our example: the universal complexification of S^1 is $(S^1)^{\mathbb{C}} = \mathbb{C}^*$). Recall that $K^{\mathbb{C}}$ is complex reductive if K is compact. In our case of interest $X = \mathbb{C}^n$ this extension always exists due to a result of FORSTNERIC ([6] 2.2 , for an extended discussion see [17]) :

The map $\pi_X : X \rightarrow X//K$ parameterizes the closed $K^{\mathbb{C}}$ -orbits in X , i.e., every π_X -fiber contains exactly one closed $K^{\mathbb{C}}$ -orbit O and moreover every G -orbit in the fiber contains O in its closure. Also the complex space $X//K \cong X//K^{\mathbb{C}}$ is stratified by the types of the closed $K^{\mathbb{C}}$ -orbits in the π_X -fibers. Here we say that two orbits $K^{\mathbb{C}} \cdot x$ and $K^{\mathbb{C}} \cdot y$ have the same type iff the isotropy groups $K_x^{\mathbb{C}}$ and $K_y^{\mathbb{C}}$ are conjugate subgroups of $K^{\mathbb{C}}$.

Last we remark that if K is a finite group the categorical quotient is the same as the orbit quotient X/K . The following easy example will be used in section 4.

EXAMPLE 2.1. Let S^1 act on \mathbb{C}^{n+4} by the rule $\lambda, (z_1, z_2, z_3, z_4, u_1, \dots, u_n) \mapsto (z_1, z_2, \lambda^2 \cdot z_3, \lambda^{-2} \cdot z_4, \lambda \cdot u_1, \dots, \lambda \cdot u_n)$, $\lambda \in S^1$, $(z_1, z_2, z_3, z_4, u_1, \dots, u_n) \in \mathbb{C}^{n+4}$. Since S^1 is an identity set in \mathbb{C}^* the S^1 -invariant holomorphic functions on \mathbb{C}^{n+4} are \mathbb{C}^* -invariant where \mathbb{C}^* acts by the same rule. Hence they are constant on \mathbb{C}^* -orbits. So one calculates that all S^1 -invariant functions are functions of $z_1, z_2, z_3 z_4$ and $z_4^2 u_i u_j$ $1 \leq i \leq j \leq n$. The map $\pi : \mathbb{C}^{n+4} \rightarrow \mathbb{C}^{n+4} // S^1$ is given by $(z_1, z_2, z_3, z_4, u_1, \dots, u_n) \mapsto (z_1, z_2, z_3 z_4, z_4^2 u_1 u_1, z_4^2 u_1 u_2 \dots, z_4^2 u_n u_n)$ and $\mathbb{C}^{n+4} // S^1$ is the image of this map in $\mathbb{C}^{\frac{(n+1)n}{2}+3}$. It is the analytic subset which can be given by

$$A := \{(b_1, b_2, b_3, b_{11}, \dots, b_{nn}) \in \mathbb{C}^{\frac{(n+1)n}{2}+3} : b_{ij} b_{kl} = b_{ik} b_{jl} \quad \forall i, j, k, l \in \{1, 2, \dots, n\}\}$$

(here we do not distinguish between b_{ij} and b_{ji}). We have 3 types of closed \mathbb{C}^* -orbits corresponding to the (isotropy) subgroups $\{1\}$, $\{\pm 1\}$ and \mathbb{C}^* (fixed points). The stratification of $A \cong \mathbb{C}^{n+4} // S^1$ is: stratum with isotropy \mathbb{C}^* in the closed orbit $-\{(b_1, b_2, b_3, b_{11}, \dots, b_{nn}) \in A : b_3 = b_{11} = \dots = b_{nn} = 0\}$; stratum with isotropy $\{\pm 1\}$ in the closed orbit $-\{(b_1, b_2, b_3, b_{11}, \dots, b_{nn}) \in A : b_{11} = \dots = b_{nn} = 0, b_3 \neq 0\}$; the stratum with trivial isotropy is the rest.

Now we formulate some positive results about holomorphic linearization. First of all we remark that a local linearization of an action of a compact Lie group K on a manifold X in a K -invariant neighborhood of a fixed point x is easily achieved by averaging over the group K (with respect to the Haar measure) a suitable local biholomorphism from a neighborhood U of x in X to a neighborhood of 0 in the tangent space $T_x X$ (see for instance [2] 2.2). For this, note that by differentiating the action map $\phi : K \times X \rightarrow X$ with respect to the second variable one gets a K -action on the tangent bundle TX hence a linear representation of K on the tangent space $T_x X$ at the fixed point x .

Moreover if the reductive group $K^{\mathbb{C}}$ acts and X is Stein one can extend this local K -equivariant biholomorphism to a $K^{\mathbb{C}}$ -equivariant biholomorphism between $K^{\mathbb{C}}$ -invariant neighborhoods, which are saturated with respect to the categorical quotient maps. This is a special case of the holomorphic version of LUNA's slice theorem ([8] 5.5).

From this one easily deduces the fact (which is classical for actions of reductive groups $K^{\mathbb{C}}$) that holomorphic actions of a compact group K on \mathbb{C}^n with zero dimensional categorical quotient are linearizable, i.e., we are speaking about the case that $\mathbb{C}^n // K$ is just one point or equivalently all K -invariant holomorphic functions on \mathbb{C}^n are constant. We remark that the above mentioned extension result of FORSTNERIC is needed to pass from $K^{\mathbb{C}}$ -linearization to K -linearization.

The first remarkable result is due to SUZUKI [22] who gave a classification of the proper holomorphic $(\mathbb{C}, +)$ -actions on \mathbb{C}^2 . Here proper means that the limit set of each \mathbb{C} -orbit is finite. \mathbb{C}^* -actions on a Stein space are proper in this sense. His result combined with FORSTNERIC's extension result yields

Theorem 2.2. *Actions of S^1 on \mathbb{C}^2 by holomorphic automorphisms are linearizable.*

The complex algebraic analogue of the linearization question for reductive groups is studied quite well. We refer the interested reader to the papers by KRAFT [13], [14]. We only remark that algebraic linearization of \mathbb{C}^* -actions on \mathbb{C}^n is open in general, but has a positive solution for $n \leq 3$ [12]. The first non-linearizable algebraic actions were found by SCHWARZ [20]. They come from non G -trivial G -vector bundles over representations. More concretely he proved that for some reductive groups G (e.g. $O_2(\mathbb{C})$) there exist algebraic actions on \mathbb{C}^n of the form

$$g \cdot (z, w) \mapsto \alpha(g) \cdot z, \phi(g, z) \cdot w, \quad g \in G, (z, w) \in \mathbb{C}^{n_1} \times \mathbb{C}^{n_2} = \mathbb{C}^n, \quad (1)$$

which are not algebraically linearizable. Here $\alpha : G \rightarrow GL_{n_1}(\mathbb{C})$ is a linear representation and $\phi : G \times \mathbb{C}^{n_1} \rightarrow GL_{n_2}(\mathbb{C})$ is an algebraic map satisfying the two conditions $\phi(e, z) = id$ and $\phi(g, \alpha(h) \cdot z) \cdot \phi(h, z) = \phi(gh, z)$, $z \in \mathbb{C}^{n_1}, g, h \in G$ to make (1) an action.

In [16] it is proved that algebraic $K^{\mathbb{C}}$ -actions with one-dimensional categorical quotient are holomorphically linearizable [16]. This result was generalized to the natural setting by JIANG [11]. Combined with FORSTNERIC's extension result one can state his result as follows

Theorem 2.3. *Actions of compact Lie groups on \mathbb{C}^n by holomorphic automorphisms with one dimensional categorical quotient are linearizable.*

Note that this result generalizes Theorem 2.2. The method of SCHWARZ to construct non-linearizable algebraic actions does not work in the holomorphic setting. A corollary of an equivariant version of Grauert's-Oka-principle proved by HEINZNER and the second author ([9]) states:

Theorem 2.4. *Actions of compact Lie groups on \mathbb{C}^n by holomorphic automorphisms of the form (1) are linearizable.*

For the last result recall that the group of overshers Sh_n is the subgroup of $\text{Aut}_{\text{hol}}(\mathbb{C}^n)$ generated by affine automorphisms and automorphisms of the form

$$(z_1, \dots, z_n) \mapsto (a(z_2, \dots, z_n)z_1 + b(z_2, \dots, z_n), z_2, \dots, z_n)$$

where a, b are arbitrary holomorphic functions on \mathbb{C}^{n-1} and a is invertible. By a theorem of ANDERSEN and LEMPERS this is a dense (but proper) subgroup of $\text{Aut}_{\text{hol}}(\mathbb{C}^n)$ $n \geq 2$ [3]. The following theorem was proved by AHERN and RUDIN in the case of finite cyclic groups [1] and generalized by KRAFT and the second author to the present form [15].

Theorem 2.5. *Every action of a compact Lie group K on \mathbb{C}^2 by elements of Sh_2 is linearizable.*

In view of Theorem 2.3 this result is only interesting for finite groups. By the methods explained in section 4 one can for instance prove that there are non-linearizable holomorphic actions of finite cyclic groups $\mathbb{Z}/n\mathbb{Z}$ on \mathbb{C}^m for $m \geq n + 2$ ([5]). The smallest dimension in which non-linearizable holomorphic actions are known is 4. So the linearization question for finite groups on \mathbb{C}^2 remains open.

3. NON-EQUIVALENT EMBEDDINGS

Definition. Two proper holomorphic embeddings $\varphi_{1,2} : \mathbb{C} \hookrightarrow \mathbb{C}^n$ are called *equivalent* if there is a holomorphic automorphism $\alpha \in \text{Aut}_{\text{hol}}(\mathbb{C}^n)$ with $\alpha(\varphi_1(\mathbb{C})) = \varphi_2(\mathbb{C})$.

REMARK 3.1. Our definition of equivalence is just a right and left equivalence with respect to automorphisms, i.e. two proper holomorphic embeddings $\varphi_1 : \mathbb{C} \rightarrow \mathbb{C}^n$ and $\varphi_2 : \mathbb{C} \rightarrow \mathbb{C}^n$ are equivalent if there exist two automorphisms $\alpha \in \text{Aut}_{\text{hol}}(\mathbb{C}^n)$ and $\beta \in \text{Aut}_{\text{hol}}(\mathbb{C})$ with $\alpha(\varphi_1(z)) = \varphi_2(\beta(z)) \quad \forall z \in \mathbb{C}$.

For the proof of the main result (3.5) of this section we need the following easy lemma. The proof is a straightforward transversality argument.

Lemma 3.2. *Let Y be a submanifold of real dimension 2 of \mathbb{C}^n and let $m > \frac{n^2+n}{n-1}$. Then the set of points $x = (x_1, \dots, x_m) \in (\mathbb{C}^n)^m$ such that there exists an affine automorphism $\alpha \in \text{Aff}(\mathbb{C}^n)$ mapping all the points x_1, \dots, x_m into Y has measure 0.*

Also we like to refer to the following two much less trivial (especially the second) results

Theorem 3.3. *Lemma 4.4. in [18]*

For each $t > 0$ and numbers $r_1 = t < r_2 < r_3 < \dots$ and $\lim_{j \rightarrow \infty} r_j = \infty$ there exists a discrete subset $E_{t, \{r_i\}} \subset \mathbb{C}^n$ lying on the union of the spheres $r_j S$ with the following property:

If $F : tB \rightarrow \mathbb{C}^n$ is a holomorphic map satisfying

- (i) $|F(0)| \leq \frac{t}{2}$
- (ii) $|(JF)(z)| \geq \frac{1}{t}$ at some point $z \in \frac{1}{2}tB$
- (iii) $F(tB) \cap E_{t, \{r_i\}} = \emptyset$

then $F(\frac{1}{2}tB) \subset tB$.

Theorem 3.4. (see [4] for a stronger interpolation result)

Given a discrete subset $E = \{e_i\}_{i \in \mathbb{N}} \subset \mathbb{C}^n$ and a discrete subset $A = \{a_i\}_{i \in \mathbb{N}} \subset \mathbb{R} \subset \mathbb{C}$ there exists a proper holomorphic embedding $\varphi : \mathbb{C} \rightarrow \mathbb{C}^n$ with $\varphi(a_i) = e_i \quad \forall i$.

Proposition 3.5. *There are uncountably many equivalence classes of proper holomorphic embeddings of \mathbb{C} into \mathbb{C}^n ($n \geq 2$).*

PROOF. Suppose there are only countably many equivalence classes of proper holomorphic embeddings of \mathbb{C} into \mathbb{C}^n . We enumerate them by natural numbers and choose one embedding φ_i from each equivalence class. We will construct a new embedding $\varphi : \mathbb{C} \rightarrow \mathbb{C}^n$ which is not equivalent to one of the φ_i 's, thus proving the proposition. For each $t \in \mathbb{N}$ we choose subsets $E_t = E_{t, \{r_i\}}$ of \mathbb{C}^n like in Theorem 3.3 which are mutually disjoint. This can be done for instance choosing the r_i 's for each t to be $r_1 = t, r_2 = t + 1 + \frac{1}{2}, \dots, r_n = t + n - 1 + \frac{1}{n}, \dots$

Let p_i be the i -th prime number and define a subset E^i of \mathbb{C}^n by:

$$E^i = \bigcup_{j=1}^{\infty} E_{p_i^j}$$

Observe that the union is disjoint and each E^i is discrete in \mathbb{C}^n . By Lemma 3.2 we find finitely many points $x_1, \dots, x_m \in \mathbb{C}^n$ with $\|x_k\| < 1 \quad \forall k$ and with the property that for all $i \in \mathbb{N}$ holds: There is no $\alpha \in \text{Aff}(\mathbb{C}^n)$ such that α maps all the points x_1, \dots, x_m into $\varphi_i(\mathbb{C})$.

Now define:

$$E = \bigcup_{i=1}^{\infty} E^i \cup \bigcup_{k=1}^m x_k$$

The set E is likewise discrete in \mathbb{C}^n and it is the disjoint union of the E^i 's and the finite number of points x_k . Let e_1, e_2, \dots be an enumeration of E beginning with $e_1 = x_1, \dots, e_m = x_m$. We will find for each $e = e_j \in E$ a point $a_j \in \mathbb{R}$ such that the a_j 's are mutually disjoint, the resulting set $A = \bigcup_{j=1}^{\infty} a_j$ is discrete in \mathbb{R} and so that a_j satisfies the following condition:

If $e_j \in E_{p_i^k}$ then

$$\|\varphi_i^{-1}(p_i^k \overline{B}) - a_j\| > (p_i^k) \cdot \|\varphi_i^{-1}(p_i^k \overline{B})\| \quad (*)$$

For that begin with $a_1 = \frac{1}{m}, a_2 = \frac{2}{m}, \dots, a_m = 1$. Further we proceed by induction. Each e_j for $j > m$ is contained exactly in one $E_{p_i^k}$. Since φ_i is proper $\varphi_i^{-1}(p_i^k \overline{B})$ is compact in \mathbb{C} hence $\|\varphi_i^{-1}(p_i^k \overline{B})\|$ is a finite number and we can choose

$$a_j > \max\{a_{j-1} + 1, ((p_i^k) + 1) \cdot \|\varphi_i^{-1}(p_i^k \overline{B})\|\}.$$

The resulting set satisfies our conditions.

Now we use Theorem 3.4 to find a proper holomorphic embedding $\varphi : \mathbb{C} \rightarrow \mathbb{C}^2$ with $\varphi(a_i) = e_i \quad \forall i \in \mathbb{N}$. We claim that φ is not equivalent to one of the φ_i 's. Suppose the contrary, i.e. there exists an $i \in \mathbb{N}$ and automorphisms $\alpha \in \text{Aut}_{\text{hol}}(\mathbb{C}^n)$ and $\beta \in \text{Aut}_{\text{hol}}(\mathbb{C})$ (lets say $\beta(z) = az + b, \quad a \in \mathbb{C}^*, b \in \mathbb{C}$) with

$$\varphi(az + b) = \alpha \circ \varphi_i(z).$$

Suppose $\alpha(p_i^k B) \cap E_{p_i^k} \neq \emptyset$. From $\alpha \circ \varphi_i(\mathbb{C}) = \varphi(\mathbb{C}) \supset E_{p_i^k}$ we conclude that $\alpha(p_i^k B \setminus \varphi_i(\mathbb{C})) \cap E_{p_i^k} = \emptyset$. So there is a point $z_k \in \varphi_i^{-1}(p_i^k \overline{B})$ with $\varphi(az_k + b) = \alpha \circ \varphi_i(z_k) \in E_{p_i^k}$. Let us denote that point by $e_{j(k)}$. Remember $\|e_{j(k)}\| > p_i^k$ by the construction of $E_{p_i^k}$. By the construction of φ we have $\varphi(a_{j(k)}) = e_{j(k)}$. Since φ is injective $a_{j(k)} = az_k + b$ and condition (*) in the construction of $A \subset \mathbb{R}$ implies:

$$\|z_k - a_{j(k)}\| > (p_i^k)\|z_k\|$$

But this is impossible for sufficiently large k because

$$(p_i^k)\|z_k\| < \|z_k - a_{j(k)}\| = \|az_k + b - z_k\| \leq \|a - 1\|\|z_k\| + \|b\|$$

forces $\lim_{k \rightarrow \infty} z_k = 0$ hence $\lim_{k \rightarrow \infty} a_{j(k)} = b$. So $e_{j(k)} = \varphi(a_{j(k)})$ would converge contradicting $\lim_{k \rightarrow \infty} \|e_{j(k)}\| = \infty$.

So we have proved that there is some k_0 such that $\alpha(p_i^k B) \cap E_{p_i^k} = \emptyset \quad k \geq k_0$. Clearly enlarging k_0 we can further assume $\|\alpha(0)\| \subset \frac{1}{2}p_i^k B$ and $\|J(\alpha)(0)\| \geq \frac{1}{p_i^k}$ for $k \geq k_0$. By the construction of $E_{p_i^k}$ (see Theorem 3.3) it follows:

$$\alpha\left(\frac{1}{2}p_i^k B\right) \subset p_i^k B \quad \forall k \geq k_0$$

This growth restriction forces α to be an affine automorphism. But no affine automorphism can map the points x_1, \dots, x_k contained in the image of φ to the image of φ_i . This contradicts the equivalence of φ and φ_i . □

4. UNCOUNTABLY MANY NON-EQUIVALENT S^1 -ACTIONS

In the proof of Proposition 4.2 we will need the fact that if the image of an embedding $\varphi : \mathbb{C}^k \hookrightarrow \mathbb{C}^n$ is contained in an linear subspace of high enough codimension this embedding is *straightenable* (by definition: There is some $\alpha \in \text{Aut}_{\text{hol}}(\mathbb{C}^n)$ with $\alpha \circ \varphi(\xi_1, \dots, \xi_k) = (\xi_1, \dots, x_k, 0, \dots, 0)$). This fact is made precise in the following lemma. For a proof we refer to [5] (Lemma 2.5).

Lemma 4.1. *If $\varphi : \mathbb{C}^k \hookrightarrow \mathbb{C}^n$ is a proper holomorphic embedding, then the embedding $\varphi_1 : \mathbb{C}^k \hookrightarrow \mathbb{C}^{n+k} = \mathbb{C}^n \times \mathbb{C}^k$ defined by $\varphi_1(z) = (\varphi(z), 0)$ is straightenable.*

For any proper holomorphic embedding $\varphi : \mathbb{C} \hookrightarrow \mathbb{C}^2$, we define a manifold X_φ in the following way: We choose a holomorphic function $f \in \mathcal{O}(\mathbb{C}^2)$ which vanishes precisely on the closed submanifold $\varphi(\mathbb{C})$ in \mathbb{C}^2 and the gradient of f does not vanish when f vanishes, in other words f is a generator of the ideal $I_{\varphi(\mathbb{C})}(\mathbb{C}^2)$ of $\varphi(\mathbb{C})$. Such a function exists by the solution of the multiplicative Cousin problem. Now we define

$$X_\varphi = \{(x, y, z, w) \in \mathbb{C}^4 : f(x, y) = w \cdot z\}.$$

Since the gradient of f does not vanish on $\{f(x, y) = 0\} = \varphi(\mathbb{C})$ the set X_φ is a smooth submanifold of \mathbb{C}^4 . Also it is clear that X_φ up to biholomorphism does not depend on the choice of the generator f . The crucial point of our construction of non-linearizable actions is the following

Proposition 4.2. *For every proper holomorphic embedding $\varphi : \mathbb{C} \rightarrow \mathbb{C}^2$ the manifold $X_\varphi \times \mathbb{C}$ is biholomorphic to \mathbb{C}^4 .*

PROOF. We have

$$X_\varphi \times \mathbb{C} = \{(x, y, z, w, u) \in \mathbb{C}^5 : f(x, y) = w \cdot z\}.$$

The map $x, y, z, w, u \mapsto x, y, w \cdot u, z, w, u$ gives a biholomorphism $\tau_1 : X_\varphi \times \mathbb{C} \rightarrow A_1$ where A_1 is the submanifold of \mathbb{C}^6 defined by

$$A_1 := \{(x, y, v, z, w, u) \in \mathbb{C}^6 : f(x, y) = w \cdot z, v = u \cdot z\}$$

The inverse mapping τ_1^{-1} is given by $(x, y, v, z, w, u) \mapsto (x, y, v, z, w, u)$.

By Lemma 4.1 there exists an automorphism $\alpha \in \text{Aut}_{\text{hol}}(\mathbb{C}^3)$ with

$$\alpha(\varphi(\xi), 0) = (\xi, 0, 0). \tag{*}$$

The automorphism of \mathbb{C}^6 defined by $(x, y, v, z, w, u) \mapsto \alpha(x, y, v), z, w, u$, i.e., in the first 3 coordinates α and identity in the other 3 coordinates gives if restricted to A_1 a biholomorphism τ_2 from A_1 to the submanifold A_2 of \mathbb{C}^6 defined by:

$$A_2 := \{(a, b, c, z, w, u) \in \mathbb{C}^6 : (\alpha^{-1})^*f(a, b, c) = w \cdot z, (\alpha^{-1})^*v(a, b, c) = u \cdot z\}$$

Because of (*) the ideal in $\mathcal{O}(\mathbb{C}^3)$ generated by the two functions b and c is the same as the ideal generated by the functions $(\alpha^{-1})^*v$ and $(\alpha^{-1})^*f$, namely the ideal $I_{\mathbb{C} \times \{0\}}$ of the first coordinate axis. We can express one set of generators through the other, that means we find holomorphic functions $a_{ij}, b_{ij} \in \mathcal{O}(\mathbb{C}^3)$ $i, j = 1, 2$ with

$$b = a_{11} \cdot (\alpha^{-1})^*f + a_{12} \cdot (\alpha^{-1})^*v, \quad c = a_{21} \cdot (\alpha^{-1})^*f + a_{22} \cdot (\alpha^{-1})^*v \tag{1}$$

and

$$(\alpha^{-1})^*f = b_{11} \cdot b + b_{12} \cdot c, \quad (\alpha^{-1})^*v = b_{21} \cdot b + b_{22} \cdot c. \tag{2}$$

We claim that the restriction τ_3 to A_3 of the holomorphic map $S_1 : \mathbb{C}^6 \rightarrow \mathbb{C}^6$ defined by

$$S_1(a, b, c, z, w, u) = a, b, c, z, a_{11}(a, b, c) \cdot w + a_{12}(a, b, c) \cdot u, a_{21}(a, b, c) \cdot w + a_{22}(a, b, c) \cdot u$$

gives a biholomorphism τ_3 from A_2 to the submanifold A_3 of \mathbb{C}^6 defined by

$$A_3 := \{(a, b, c, z, w, u) \in \mathbb{C}^6 : b = w \cdot z, c = u \cdot z\}.$$

We will prove that the inverse $\tau_3^{-1} : A_3 \rightarrow A_2$ is given by the restriction of $S_2 : \mathbb{C}^6 \rightarrow \mathbb{C}^6$ defined by

$$S_2(a, b, c, z, w, u) = a, b, c, z, b_{11}(a, b, c) \cdot w + b_{12}(a, b, c) \cdot u, b_{21}(a, b, c) \cdot w + b_{22}(a, b, c) \cdot u$$

to A_3 . For that consider the holomorphic maps

$$\psi_1 : \mathbb{C}^3 \times \mathbb{C}^* \rightarrow \mathbb{C}^6, \quad a, b, c, z \mapsto a, b, c, z, \frac{(\alpha^{-1})^*f}{z}, \frac{(\alpha^{-1})^*v}{z}$$

and

$$\psi_2 : \mathbb{C}^3 \times \mathbb{C}^* \rightarrow \mathbb{C}^6, \quad a, b, c, z \mapsto a, b, c, z, \frac{b}{z}, \frac{c}{z}.$$

The submanifold A_2 is the topological closure (and therefore the holomorphic closure, i.e., the smallest analytic set containing ...) of the image of ψ_1 in \mathbb{C}^6 . Also A_3 is the closure of the image of ψ_2 in \mathbb{C}^6 .

From (1) follows $\psi_2 = S_1 \circ \psi_1$ and (2) implies $\psi_1 = S_2 \circ \psi_2$. So we have $\psi_1 = S_2 \circ S_1 \circ \psi_1$. This means that $S_1 \circ S_2$ is the identity on the image of ψ_1 hence it is the identity on the closure of the image of ψ_1 , on A_2 . Analogously follows that $S_2 \circ S_1$ is the identity on A_3 .

Finally the map $\tau_4 : A_3 \rightarrow \mathbb{C}^4$, $(a, b, c, z, w, u) \mapsto a, z, w, u$ is a biholomorphism. The composition $\tau_4 \circ \tau_3 \circ \tau_2 \circ \tau_1$ provides the desired biholomorphism from $X_\varphi \times \mathbb{C}$ to \mathbb{C}^4 .

□

REMARK 4.3. We do not know whether the manifold X_φ itself is biholomorphic to \mathbb{C}^3 (if $\varphi : \mathbb{C} \hookrightarrow \mathbb{C}^2$ is straightenable then this is clearly true). If for some non-straightenable φ the manifold X_φ would be biholomorphic to \mathbb{C}^3 , then we would have a non-linearizable S^1 -action on $X_\varphi \cong \mathbb{C}^3$ (see the proof of Proposition 4.5 below), if it is not biholomorphic to \mathbb{C}^3 then this would be a counterexample to the following open problem.

Problem 4.4 (Holomorphic Zariski Cancellation Problem). *Let Z be a complex manifold such that $Z \times \mathbb{C}$ is biholomorphic to \mathbb{C}^{n+1} ($n \geq 2$). Does it follow that $Z \cong \mathbb{C}^n$?*

Lemma 4.5. *If $\varphi_{1,2} : \mathbb{C} \rightarrow \mathbb{C}^n$ are two non-equivalent proper holomorphic embeddings, then for all $k \geq 1$ the actions of S^1 on $X_{\varphi_i} \times \mathbb{C}^k$ $i = 1, 2$ given by*

$$S^1 \times X_{\varphi_i} \times \mathbb{C}^k \rightarrow X_{\varphi_i} \times \mathbb{C}^k$$

$$\lambda, z_1, z_2, z_3, z_4, u_1, \dots, u_k \mapsto z_1, z_2, (\lambda)^2 z_3, (\lambda)^{-2} z_4, \lambda u_1, \dots, \lambda u_k$$

are not conjugated by a holomorphic automorphism of \mathbb{C}^{k+3} .

PROOF. An equivariant biholomorphism between $X_{\varphi_1} \times \mathbb{C}^k$ and $X_{\varphi_2} \times \mathbb{C}^k$ would imply a biholomorphism of the categorical quotients respecting the stratification by orbit types. The categorical quotients of these two S^1 -manifolds are the restrictions of the categorical quotient maps of \mathbb{C}^{4+k} from example 2.1 to the submanifolds $X_{\varphi_i} \times \mathbb{C}^k$ (this is a general property of the categorical quotient of a Stein K -space). One can easily check that the complement of the stratum with trivial isotropy is in both cases biholomorphic to \mathbb{C}^2 in which the fixed point stratum (which is biholomorphic to \mathbb{C}) is embedded like $\varphi_i(\mathbb{C}) \subset \mathbb{C}^2$. Contradiction. \square

Corollary 4.6. *For $n \geq 4$ there are uncountably many conjugacy classes of subgroups of $\text{Aut}_{\text{hol}}(\mathbb{C}^n)$ isomorphic to S^1 .*

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