

# Subvarieties of $\mathbb{C}^n$ with non-extendable automorphisms

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**Abstract.** We investigate algebraic and analytic subvarieties of  $\mathbb{C}^n$  with automorphisms which cannot be extended to the ambient space.

## 1. Introduction

If  $Z$  is an affine variety and  $n$  is a sufficiently large number, then any two embeddings  $i, j: Z \rightarrow \mathbb{A}^n$  are equivalent in the sense that there exists an automorphism  $\phi$  of  $\mathbb{A}^n$  such that  $\phi \circ i = j$  (see [14], [15], [20]). In particular, in this case every automorphism of  $Z$  extends to the whole of  $\mathbb{A}^n$ . In the case where  $Z$  is smooth it suffices to take  $n > 2 \dim Z + 1$ .

Thus for an algebraic subvariety  $Z \subset \mathbb{A}^n$  of high codimension every automorphism extends to an automorphism of  $\mathbb{A}^n$ .

This raises three questions:

1. Is a similar statement true for subvarieties of low codimension, e.g. hypersurfaces?
2. Does a similar statement hold in the holomorphic category?
3. Assume that  $Z \subset \mathbb{A}^n$  is a subvariety such that every single automorphism of  $Z$  extends to  $\mathbb{A}^n$ . Does this imply that there is an extension of the action of the group  $\text{Aut}(Z)$  to an action on  $\mathbb{A}^n$ ?

The purpose of this article is to provide negative answers to all three questions.

First, we prove that there exist irreducible algebraic hypersurfaces admitting a  $(\mathbb{C}, +)$ -action which does not extend.

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**Theorem 1.** *There exists a smooth irreducible algebraic hypersurface  $H \subset \mathbb{A}^5$  and an algebraic action  $\mu$  of the additive group  $(\mathbb{C}, +)$  on  $H$  such that for all  $t \neq 0$  there exists neither an algebraic nor a holomorphic automorphism  $\phi$  of  $\mathbb{A}^5$  with  $\phi|_H = \mu(t)$ .*

The key idea for our construction is to choose a hypersurface in  $\mathbb{A}^n$  such that the complement has a small automorphism group and then to construct an automorphism on the hypersurface based on something which does not extend from the hypersurface to the whole space. For this it is crucial that there are free actions of algebraic groups  $G$  on the affine space  $\mathbb{A}^n$  for which there exist  $G$ -invariant subvarieties  $W$  such that the restriction map  $\mathcal{O}(\mathbb{A}^n)^G \rightarrow \mathcal{O}(W)^G$  is not surjective ([21]).

It should be noted that this strategy cannot work for subvarieties of higher codimension on  $\mathbb{A}^n$ , because for a subvariety  $Z$  in  $\mathbb{A}^n$  of codimension at least two the automorphism group of the complement is always quite large. In fact the group of all automorphisms of  $\mathbb{A}^n$  fixing every point in  $Z$  acts *transitively* on the complement  $\mathbb{A}^n \setminus Z$  as soon as  $\text{codim}(Z) \geq 2$  (see [22]).

In the second part, we prove that there does not exist any effective differentiable, holomorphic or algebraic action of the group  $\text{Aut}(\mathbb{C}^* \times \mathbb{C}^*)$  on the affine space  $\mathbb{C}^n$ .

**Theorem 2.** *Let  $k$  be an algebraically closed field,  $K$  a field,  $n \in \mathbb{N}$  and let  $G$  denote the group given by the semidirect product  $\text{SL}_2(\mathbb{Z}) \rtimes_{\varrho} (k^* \times k^*)$  with*

$$\varrho \begin{pmatrix} a & b \\ c & d \end{pmatrix} (z, w) = (z^a w^b, z^c w^d).$$

*Then there does not exist any injective group homomorphism from  $G$  either into the group  $\text{Diff}(\mathbb{R}^n)$  of diffeomorphisms of  $\mathbb{R}^n$  or into the group  $\text{Aut}_K(\mathbb{A}^n)$  of  $K$ -automorphisms of the affine space  $\mathbb{A}^n$ .*

Hence, whenever  $i: Z = \mathbb{C}^* \times \mathbb{C}^* \hookrightarrow \mathbb{C}^n$  has the property that every automorphism of  $Z$  extends to  $\mathbb{C}^n$ , there must be a non-trivial non-split short exact sequence of groups

$$1 \rightarrow L \rightarrow \tilde{G} \xrightarrow{\alpha} \text{Aut}(Z) \rightarrow 1$$

such that for every  $g \in \tilde{G}$  the restriction  $\mu(g)|_Z$  coincides with the canonical action of  $\alpha(g) \in \text{Aut}(Z)$  on  $i(Z)$ .

Finally, in the third section, we discuss the situation in the analytic category. Unlike the algebraic situation there are analytic subvarieties of high codimension such that no non-trivial automorphism extends.

The first such examples are due to Rosay and Rudin [16] who proved that for  $n \geq 2$  there exist *discrete* subsets  $S$  of  $\mathbb{C}^n$  such that no non-trivial permutation of  $S$  extends to an automorphism of  $\mathbb{C}^n$ .

Buzzard and Fornæss [4] proved that given a hypersurface  $X$  in  $\mathbb{C}^N$  there exists an embedding  $j: X \rightarrow \mathbb{C}^N$  such that the complement  $\mathbb{C}^N \setminus X$  is hyperbolic. This implies that at most countably many automorphisms of  $X$  can be extended to automorphisms of  $\mathbb{C}^N$ .

Thus the existence of analytic subvarieties with non-extendable automorphisms is well-known for proper analytic subvarieties of maximal or minimal codimension.

Our contribution is to provide a result about analytic subvarieties of arbitrary codimension.

We prove the following:

**Theorem 3.** *For every non-finite analytic subvariety  $X \subset \mathbb{C}^n$ , every Lie group  $G$  and every effective  $G$ -action on  $X$  there exists an embedding  $j: X \rightarrow \mathbb{C}^n$  such that for no element  $g \in G \setminus \{e\}$  the induced automorphism of  $j(X)$  can be extended to an automorphism of  $\mathbb{C}^n$ .*

As a consequence we obtain the following result:

**Corollary 1.** *Let  $X$  be an infinite Stein manifold such that  $\text{Aut}(X)$  is a Lie group. Then there exists an embedding  $j$  of  $X$  into some  $\mathbb{C}^N$  such that no non-trivial automorphism of  $X$  extends to  $\mathbb{C}^N$ .*

The condition that  $\text{Aut}(X)$  is a Lie group holds in particular if  $X$  is hyperbolic, e.g. if the universal covering of  $X$  is biholomorphic to a bounded domain.

## 2. Hypersurfaces

In this section we investigate the algebraic subvarieties of low codimension, concentrating on hypersurfaces. Varieties, functions, maps, group actions etc. are assumed to be algebraic over some algebraically closed ground field  $k$ . Unless explicitly stated otherwise, this ground field  $k$  may have positive characteristic.

**2.1. Basic tools.** We start with some basic observations which will be used in later constructions.

**Lemma 1.** *Let  $\pi: X \rightarrow Y$  be a separable surjective morphism between irreducible normal algebraic varieties defined over some ground field  $k$ . Let  $\phi \in \text{Aut}_k(X)$  and assume that  $\pi \circ \phi|_F$  is constant for every  $\pi$ -fiber  $F$ .*

*Then  $\phi$  induces a  $k$ -automorphism  $\phi'$  of  $Y$  such that  $\pi \circ \phi = \phi' \circ \pi$ .*

*Proof.* Since  $\pi$  is separable and dominant, the function field  $k(Y)$  is isomorphic to the subfield  $L$  of  $k(X)$  containing all those rational functions on  $X$  which are constant along the  $\pi$ -fibers. Hence  $\phi$  induces an automorphism of the function field  $k(Y)$ . On the other hand,  $\phi$  induces a permutation of the points of  $Y$ , because  $\pi$  is surjective, and  $\pi \circ \phi$  is constant along the  $\pi$ -fibers. Since  $Y$  is normal, a rational function  $f$  on  $Y$  is regular in a given point  $y$  if and only if  $f$  has no pole in  $y$ . Using this fact, it follows that  $\phi$  induces a regular automorphism of  $Y$ .  $\square$

**Lemma 2.** *Let  $H$  be an irreducible hypersurface in the affine space  $\mathbb{A}^n$  defined by a (regular) function  $f$  and  $\Omega = \mathbb{A}^n \setminus H$ .*

*Then the map  $f|_{\Omega}: \Omega \rightarrow \mathbb{A}^1 \setminus \{0\}$  is equivariant for the whole (algebraic) automorphism group  $\text{Aut}(\Omega)$ .*

*Proof.* Let  $g \in k[\Omega]^*$  and let  $n$  denote the multiplicity of  $g$  along  $H$ . Then  $gf^{-n} \in k[\mathbb{A}^n]^* = k^*$ , hence  $g = \alpha f^n$  for some constant  $\alpha \in k^*$ . It follows that two points  $p, q \in \Omega$  are in the same  $f$ -fiber if and only if  $g(p) = g(q)$  for all  $g \in k[\Omega]^*$ . Thus the equivalence relation on  $\Omega$  defined by  $f$  is natural and must be preserved by all automorphisms of  $\Omega$ . Thus it follows from lemma 1 that every  $\phi \in \text{Aut}(\Omega)$  induces an automorphism  $f_*\phi$  of  $\mathbb{A}^1 \setminus \{0\}$  such that  $f \circ \phi = (f_*\phi) \circ f$ .  $\square$

**Lemma 3.** *Let  $P$  be a polynomial automorphism of  $\mathbb{A}^2$  stabilizing the set*

$$C = \{(x, y) : xy = 0\}.$$

*Then either  $P(x, y) = (\alpha x, \beta y)$  or  $P(x, y) = (\alpha y, \beta x)$  (with  $\alpha, \beta \in k^*$ ).*

*Proof.* Let  $\tau(x, y) = (y, x)$ . Then either  $P$  or  $P \circ \tau$  stabilize both irreducible components of  $C$ . By the preceding lemma this implies that  $P$  resp.  $P \circ \tau$  is simultaneously equivariant for both projections  $(x, y) \mapsto x$  and  $(x, y) \mapsto y$ . From this fact the statement is easily deduced.  $\square$

**2.2. Basic examples.** Here we collect some basic examples which are not irreducible.

**Example 1.** Let  $k$  be a field,  $z \in k \setminus \{0, 1, -1\}$  and  $S = \{0, 1, z\}$ . Then

$$\sigma(0) = 0, \quad \sigma(1) = z, \quad \sigma(z) = 1$$

defines a permutation of  $S$  which does not extend to an automorphism of  $\mathbb{A}^1$ .

*Proof.* This is immediate, since every automorphism of  $\mathbb{A}^1$  is affine-linear.  $\square$

**Example 2.** Let  $C \subset \mathbb{A}^2$  be the reducible curve defined by

$$C = \{(x, y) : x(xy - 1) = 0\}.$$

The action of the additive group  $G_a$  given by  $\mu_t(x, y) = (x, y)$  for  $xy = 1$  and

$$\mu_t(0, y) = (0, y + t)$$

cannot be extended to an algebraic  $G_a$ -action on  $\mathbb{A}^2$  although there does exist a holomorphic extension in the case  $k = \mathbb{C}$ .

*Proof.* Note that the two irreducible components of  $C$  are non-isomorphic. Hence every automorphism of  $\mathbb{A}^2$  stabilizing  $C$  must stabilize both irreducible components separately. It follows that such an algebraic automorphism must be equivariant for both

$(x, y) \mapsto x$  and  $(x, y) \mapsto xy$ . From this it is easily deduced that any such automorphism is of the form  $(x, y) \mapsto \left(\alpha x, \frac{1}{\alpha} y\right)$  for some  $\alpha \in k^*$ .

But an automorphism of this form cannot restrict to  $(0, y) \mapsto (0, y + t)$  on  $\{(x, y) : x = 0\}$ .

On the other hand, a holomorphic extension is given by

$$t : (x, y) \mapsto \left(x, ye^{-tx} - \frac{e^{-tx} - 1}{x}\right). \quad \square$$

**Example 3.** Let  $C \subset \mathbb{A}^2$  be the reducible curve defined by

$$C = \{(x, y) : xy(xy - 1) = 0\}.$$

Let  $\phi_0$  denote the algebraic automorphism of  $C$  given by  $\phi_0(x, y) = (y, x)$  for  $xy = 1$  and  $\phi_0(x, y) = (x, y)$  for  $xy = 0$ .

For  $k = \mathbb{C}$  there does not exist a homeomorphism  $\phi$  of  $\mathbb{A}^2(k) = \mathbb{C}^2$  with  $\phi|_C = \phi_0$ .

*Proof.* Let  $D = \{(x, y) : xy = 0\}$  and  $\Omega = \mathbb{C}^2 \setminus D$ . Consider  $\zeta_\varepsilon : S^1 \rightarrow \Omega$  given by  $\zeta_\varepsilon(z) = (z, \varepsilon)$  (with  $\varepsilon \neq 0$ ). If  $\phi$  is a homeomorphism of  $\mathbb{C}^2$  with  $\phi|_D = \text{id}$ , then  $\lim_{\varepsilon \rightarrow 0} \zeta_\varepsilon = (\text{id}, 0)$  implies that  $\phi$  stabilizes the element of  $\pi_1(\Omega)$  corresponding to  $\zeta_\varepsilon$ . Now  $\pi_1(\Omega) \simeq \mathbb{Z}^2$  is generated by this element and similar curves around the  $\{y = 0\}$ -line. This implies that such a homeomorphism  $\phi$  must induce the identity map on  $\pi_1(\Omega)$ . Since  $\pi_1(Q)$  embeds into  $\pi_1(\Omega)$  for  $Q = \{(x, y) : xy = 1\}$ , it follows that  $\phi|_Q$  must induce the identity map on  $\pi_1(Q)$ . Therefore  $\phi|_Q$  can not equal the map  $(x, y) \mapsto (y, x)$  on  $Q$  for any homeomorphism  $\phi : \mathbb{C}^2 \rightarrow \mathbb{C}^2$  with  $\phi|_D = \text{id}_D$ .  $\square$

**Example 4.** Let  $(n, m) \neq (1, 1)$  be a pair of coprime natural numbers,

$$C = \{(x, y) : x^n y^m = 1\}$$

and  $\phi \in \text{Aut}(C)$  be given by  $\phi(z, w) = (1/z, 1/w)$ .

Then  $C$  is a smooth connected curve and  $\phi$  is an automorphism of  $C$  which cannot be extended to an automorphism of  $\mathbb{A}^2$ .

*Proof.* Due to lemma 2 the map  $(x, y) \mapsto x^n y^m - 1$  is equivariant for every automorphism  $\phi \in \text{Aut}(\mathbb{A}^2)$  which stabilizes  $C$ . In particular such an automorphism stabilizes the set  $\{x^n y^m = 0\} = \{xy = 0\}$ , since this is the only reducible fiber of  $(x, y) \mapsto x^n y^m - 1$ . Hence lemma 3 implies that any automorphism of  $\mathbb{A}^2$  stabilizing  $C$  must be described as either  $(\alpha x, \beta y)$  or  $(\alpha y, \beta x)$ . The second type does not stabilize  $C$  and the former does not restrict to  $\phi$ .  $\square$

**Example 5.** Let  $H = \{(x, y, z) : xy = 1\}$  and  $\phi : (x, y, z) \mapsto (x, y, xz)$ . Then  $\phi|_H$  is an automorphism of  $H$  which cannot be extended to an automorphism of  $\mathbb{A}^3$ .

*Proof.* Assume that  $\phi|_H$  extends to a polynomial automorphism  $P$  of  $\mathbb{A}^3$ . Let  $\Omega = \{(x, y, z) : xy \neq 1\}$  and  $U = \{(x, y) : xy \neq 1\}$  and consider the morphisms

$$\Omega \xrightarrow{\pi} U \xrightarrow{e} \mathbb{A}^*$$

given by  $\pi(x, y, z) = (x, y)$  and  $e(x, y) = xy - 1$ . By lemma 2 both  $e$  and  $e \circ \pi$  are equivariant for all automorphisms of  $U$  resp.  $\Omega$ . Now all the  $\pi$ -fibers are lines and the generic  $e$ -fiber is  $\mathbb{A}^*$ . Since every morphism from  $\mathbb{A}$  to  $\mathbb{A}^*$  is constant, it follows (with the help of lemma 1) that  $\pi$  is equivariant as well.

The map  $e$  being equivariant implies in particular that the special fiber  $e^{-1}(-1)$  is invariant under all automorphisms  $\mathbb{A}^2$  stabilizing  $U$ . It follows that any automorphism  $Q$  of  $\mathbb{A}^2$  stabilizing  $U$  must also stabilize  $\{(x, y) \in \mathbb{A}^2 : xy = 0\}$ . Lemma 3 thus implies that there exist  $\alpha, \beta \in K$  such that either  $Q(x, y) = (\alpha x, \beta y)$  or  $Q(x, y) = (\beta y, \alpha x)$ .

However,  $Q$  is supposed to fix  $\{xy = 1\}$  pointwise. This forces  $\alpha = \beta = 1$  and  $Q = \text{id}_{\mathbb{A}^2}$ .

Thus an algebraic automorphism  $P$  of  $\mathbb{A}^3$  extending  $\phi|_H$  can only act along the  $\pi$ -fibers and therefore can be written in the form

$$P : (x, y, z) \mapsto (x, y, g(x, y, z))$$

for some function  $g$ . The determinant of the Jacobian of such a map equals  $\frac{\partial g}{\partial z}$ . As a nowhere vanishing regular function it must be constant. This implies that  $g$  can be written in the form  $g(x, y, z) = g_0(x, y, z^p) + \alpha z$  with  $\alpha \in k^*$ ,  $p = \text{char}(k)$  and  $g_0 \in k[X_1, X_2, X_3]$ . But now  $g|_{\{xy=1\}}$  cannot equal  $xz$  and hence we obtained a contradiction to the assumption that  $\phi_H$  extends to an algebraic automorphism  $P$  of  $\mathbb{A}^3$ .  $\square$

### 2.3. Main hypersurface example.

**Theorem 1.** *There exists a smooth irreducible hypersurface  $H \subset \mathbb{A}^5$  and an algebraic action  $\mu$  of the additive group  $(\mathbb{C}, +)$  on  $H$  such that for all  $t \neq 0$  there exists neither an algebraic nor a holomorphic automorphism  $\phi$  of  $\mathbb{A}^5$  with  $\phi|_H = \mu(t)$ .*

*Proof.* In [21] it is shown that there exists an algebraic  $\mathbb{C}$ -principal bundle  $\pi : \mathbb{C}^5 \rightarrow X$  with  $X = Q \setminus (S \cup E)$ , where  $Q$  is a smooth projective quadric,  $S$  is a smooth hypersurface and  $E$  a two-dimensional smooth subvariety of  $Q$  which intersects  $S$  transversally. Let  $Q_1$  denote the variety obtained by  $Q$  blowing-up  $E$ . We may, if necessary, blow-up  $Q_1$  again and thereby assume that there is a projective manifold  $\hat{Q}$ , a divisor  $D$  with simple normal crossings as its only singularities, an irreducible component  $D_0$  of  $D$  and a birational connected surjective morphism  $\tau : \hat{Q} \rightarrow Q$  with  $\tau(D_0) = E$  inducing an isomorphism  $\hat{Q} \setminus D \simeq Q \setminus (S \cup E)$ . Let  $F$  be a generic fiber of  $\tau|_{D_0} \rightarrow E$  (i.e.  $F$  is smooth and connected). Now we fix a very ample line bundle  $\mathcal{O}(1)$  on  $\hat{Q}$  and choose  $n \in \mathbb{N}$  such that  $\mathcal{O}(n) \otimes K_{\hat{Q}}$  is ample. By Bertini's theorem, for every smooth submanifold  $Z \subset \hat{Q}$  there is a dense open subset in the linear system  $|\mathcal{O}(n)|$  such that every divisor therein intersects  $Z$  transversally. Let  $(D_i)_{i \in I}$  be the family of irreducible components of  $D$ . Applying Bertini's theorem to all  $\bigcap_{j \in J} D_j$  with  $J \subset I$ ,  $F$  and  $\hat{Q}$  itself we may conclude that there is a very ample divisor  $H$

on  $\hat{Q}$  such that  $D \cup H$  is again a divisor with only simple normal crossings as singularities and furthermore such that  $H$  intersects  $F$  transversally. Let  $H_0 = \tau(H) \subset Q$ . Since  $H$  intersects  $F$  transversally, it is clear that  $E \subset H_0$ . Now  $H_0 \setminus S$  is affine and  $E \setminus S$  is a hypersurface in  $H_0$ . It follows that there exist rational functions on  $H_0$  which are regular on  $H_0 \setminus (E \cup S)$  and have poles of arbitrarily high multiplicity on  $E \setminus S$ . Fix such a function  $f$  of a sufficiently high pole order. We will see later, at the end of the proof, how high this multiplicity has to be. Now let  $\varrho: \mathbb{C} \times \mathbb{C}^5 \rightarrow \mathbb{C}^5$  denote the principal right action on the  $\mathbb{C}$ -principal bundle  $\pi: \mathbb{C}^5 \rightarrow X$  and let  $Y = \pi^{-1}(H_0)$ . We define a  $\mathbb{C}$ -action on  $Y$  by

$$\mu_f(t): y \mapsto \varrho(t \cdot f(\pi(y)), y)$$

and claim that for  $t \neq 0$  the automorphism  $\mu_f(t)$  of the hypersurface  $Y$  cannot be extended to an automorphism of  $\mathbb{C}^5$ , neither algebraically nor holomorphically.

Fix some  $t \neq 0$  and assume that  $\phi \in \text{Aut}_{\text{hol}}(\mathbb{C}^5)$  is an extension of  $\mu_f(t)$ . There is no loss in generality in assuming  $t = 1$ . Then  $\phi$  stabilizes  $\mathbb{C}^5 \setminus Y$  and therefore induces a holomorphic map  $\pi \circ \phi: \mathbb{C}^5 \setminus Y \rightarrow X \setminus H_0$ . We recall that  $X \setminus H_0$  embeds into  $\hat{Q}$  in such a way that the complement is  $D \cup H$ ,  $D \cup H$  is a divisor with simple normal crossings and  $D + H + K$  is ample on  $\hat{Q}$ . By a theorem of Griffiths and King ([11], Prop. 8.8) this implies that  $\pi \circ \phi$  is algebraic. Furthermore  $X \setminus H_0$  is of log general type as defined by Iitaka [13]. We claim that  $\phi$  must map  $\pi$ -fibers into  $\pi$ -fibers. Indeed, otherwise there would exist an irreducible algebraic subvariety  $R$  of codimension two in  $\mathbb{A}^5$  such that there is a dominant morphism  $F: R \times \mathbb{C} \rightarrow X \setminus H_0$  given by

$$F(r, t) = \pi \circ \phi(\varrho(t, r)).$$

Since  $X \setminus H_0$  is of log general type, such a map cannot exist. It follows that  $\pi$  is equivariant for the automorphism  $\phi$ . However, being of log general type  $X \setminus H_0$  admits only finitely many automorphisms [13], [18]. Therefore there is a number  $m$  such that  $\phi^m$  induces the trivial action on the base, i.e.  $\phi^m$  acts only along the  $\pi$ -fibers.

Let  $(U_i)_{i \in K}$  denote a covering of  $X$  by open affine subsets. The  $\mathbb{C}$ -principal bundle may be described in terms of transition functions  $\zeta_{ij} \in \mathbb{C}[U_i \cap U_j]$ . With respect to the corresponding local trivialization  $\psi = \phi^m$  is given by  $(p, x) \mapsto (p, \alpha_i(p)x + \beta_i(p))$  for  $(p, x) \in U_i \times \mathbb{A}^1$  with  $\alpha_i \in \mathcal{O}^*(U_i)$  and  $\beta_i \in \mathcal{O}(U_i)$ . An easy calculation shows that  $\alpha_i = \alpha_j$  and  $\beta_i = \beta_j + (\alpha_i - 1)\zeta_{ij}$  on  $U_i \cap U_j$ . Thus  $\alpha = \alpha_i$  is a global holomorphic function defined on the whole  $X$ . Since  $\text{codim}(E) = 2$ , the function  $\alpha$  can actually be defined on the affine variety  $Q^* = Q \setminus S$ . Recall that the transition functions  $\zeta_{ij}$  are algebraic functions on  $U_{ij}$  and therefore extend to rational functions on the whole  $Q$ . It follows that  $(\alpha - 1)\zeta_{ij}$  can be extended to a meromorphic function on  $Q$  for all  $i, j$ . Using  $\beta_i = \beta_j + (\alpha - 1)\zeta_{ij}$ , this implies that the functions  $\beta_i$  can be extended to meromorphic functions on  $Q^* = \bigcup_i U_i$ .

Moreover, if  $h$  is a regular function on  $Q^*$  such that none of the functions  $h\zeta_{ij}$  has poles on  $Q^*$ , then the functions  $h\beta_i$  haven't any poles in  $Q^*$  either.

Now observe that  $\alpha|_{H_0 \cap Q^*} \equiv 1$  and  $\beta_i|_{H_0 \cap Q^*} \equiv mf$  for all  $i$ , because  $\psi = \phi^m$  coincides with  $\mu_f(m)$  on  $\pi^{-1}(H_0 \cap Q^*)$ . It follows that, for  $h$  chosen as above,  $hf$  defines a holomorphic function on  $(H_0 \cap Q^*) \cup E$ . However, the condition which  $h$  has to fulfill (namely, that all the  $h\zeta_{ij}$  are regular on  $Q^*$ ) does not depend on the choice of  $f$ . Hence it is possible to

choose  $f$  and  $h$  in such a way that  $hf$  has poles in  $E$  and in this case the assumption of the existence of  $\phi$  leads to a contradiction.

Thus it is possible to choose  $f$  such that the resulting group action  $\mu_f(t)$  on the hypersurface  $Y$  in  $\mathbb{C}^5$  cannot be extended to an algebraic or holomorphic automorphism of  $\mathbb{C}^5$ .  $\square$

**Remark.** If one is interested only in the algebraic non-extendibility, then the last steps of the proof can be simplified substantially:

If  $\alpha$  is algebraic, it is necessarily constant. This implies  $\alpha \equiv 1$ , since  $\alpha|_{H_0 \cap Q^*} \equiv 1$ . Hence  $\beta = \beta_i$  is a global regular function and the non-extendibility of  $\mu_f(t)$  follows directly from the fact that  $f$  does not extend to a regular function on  $X$ .

**Remark.** Every  $\mathbb{C}$ -principal bundle over a differentiable manifold is differentially trivial. Using this fact, it is easy to see that the  $\mathbb{C}$ -action on  $H$  does extend to a differentiable action on  $\mathbb{C}^5$ .

### 3. Extending actions of whole groups

As mentioned above, every affine variety  $Z$  may be embedded into some affine space  $\mathbb{A}^n$  in such a way that every automorphism of  $Z$  extends to an automorphism of  $\mathbb{A}^n$ . If  $\text{Aut}(\mathbb{A}^n, Z)$  denotes the group of all automorphisms of  $\mathbb{A}^n$  stabilizing  $Z$  (as a set, not pointwise), this is equivalent to the assertion that the natural group homomorphism  $\text{Aut}(\mathbb{A}^n, Z) \rightarrow \text{Aut}(Z)$  is surjective. Thus there exists a short exact sequence

$$1 \rightarrow L \rightarrow \text{Aut}(\mathbb{A}^n, Z) \rightarrow \text{Aut}(Z) \rightarrow 1$$

one can ask if it splits.

Here we will show that there is an affine variety, namely  $Z = \mathbb{A}_1^* \times \mathbb{A}_1^*$ , such that this sequence never splits. More precisely we will prove the following theorem.

**Theorem 2.** *Let  $k$  be an algebraically closed field,  $K$  a field,  $n \in \mathbb{N}$  and let  $G$  denote the group given by the semidirect product  $\text{SL}_2(Z) \rtimes_{\varrho} (k^* \times k^*)$  with*

$$\varrho \begin{pmatrix} a & b \\ c & d \end{pmatrix} (z, w) = (z^a w^b, z^c w^d).$$

*Then there does not exist any injective group homomorphism from  $G$  either into the group  $\text{Diff}(\mathbb{R}^n)$  of diffeomorphisms of  $\mathbb{R}^n$  or into the group  $\text{Aut}_K(\mathbb{A}^n)$  of  $K$ -automorphisms of the affine space  $\mathbb{A}^n$ .*

**Corollary 2.** *Let either  $Z$  denote the complex manifold  $\mathbb{C}^* \times \mathbb{C}^*$  and  $H$  be the group of holomorphic automorphisms of  $Z$  or let  $k$  be an algebraically closed field,*

$$Z = \mathbb{A}^2 \setminus \{(z_1, z_2) : z_1 z_2 = 0\}$$

and  $H$  denote the group of  $k$ -automorphisms of  $Z$ .

Then for every holomorphic resp. regular  $H$ -action on  $\mathbb{C}^n$  resp.  $\mathbb{A}^n$  there does not exist any non-constant  $H$ -equivariant holomorphic resp. regular map from  $Z$  to  $\mathbb{C}^n$  resp.  $\mathbb{A}^n$ .

*Proof.* Thanks to the theorem there is no injective group homomorphism from  $G$  into  $\text{Aut}_{\text{hol}}(\mathbb{C}^n)$  resp.  $\text{Aut}_k(\mathbb{A}^n)$ . Since  $G \subset H$ , it follows that an equivariant map  $F$  must have non-trivial fibers such that a non-trivial normal subgroup of  $G$  acts only in fiber direction. Now  $Z$  is homogeneous under the  $G$ -action and it is easy to check that every non-trivial  $G$ -equivariant fibration of  $Z$  is of the form  $\tau_N : Z \rightarrow Z$  with

$$\tau_N : (z_1, z_2) \mapsto (z_1^N, z_2^N).$$

In this case the kernel of the induced group homomorphism  $F_*$  from  $G$  to  $\text{Aut}_{\text{hol}}(\mathbb{C}^n)$  resp.  $\text{Aut}_k(\mathbb{A}^n)$  is just

$$K_N = \{(e, \omega_1, \omega_2) : \omega_i^N = 1\} \subset \text{SL}_2(\mathbb{Z}) \ltimes_{\rho} (k^* \times k^*).$$

But  $G/K_N \simeq G$  for every  $N \in \mathbb{N}$ . Thus there is no injective group homomorphism from  $G/K_N$  into  $\text{Diff}(\mathbb{R}^n)$  or  $\text{Aut}_k(\mathbb{A}^n)$  and therefore there is no non-constant equivariant holomorphic map resp. morphism from  $Z$  to  $\mathbb{C}^n$  resp.  $\mathbb{A}^n$ .  $\square$

As a first step towards theorem 2 we need the following well-known result on the existence of fixed points.

**Proposition 1.** *Let  $p$  be a prime and let  $\Gamma$  be a finite abelian  $p$ -group acting differentiable on  $\mathbb{R}^n$  or algebraically on an affine space  $\mathbb{A}^n$  defined over a field  $k$  with  $\text{char}(k) \neq p$ .*

*Then  $\Gamma$  has a fixed point.*

*Proof.* In the first case this follows from Smith theory [12], in the second case a proof may be sketched as follows: Every  $\gamma \in \Gamma$  induces an automorphism of  $\mathbb{A}^n$  given by a polynomial map. Let  $R \subset k$  be the ring generated by all the coefficients of these polynomials for all  $\gamma \in \Gamma$ . Let  $\mathfrak{p}$  be a prime ideal in  $R$  and consider reduction modulo  $\mathfrak{p}$ . Then  $R/\mathfrak{p}$  is a finite field. If  $\text{char}(R/\mathfrak{p}) \neq p$ , then  $p$  does not divide the number of points in  $\mathbb{A}^n(R/\mathfrak{p})$ . This implies that there must be a fixed point modulo  $\mathfrak{p}$ . Finally, the existence of fixed points for almost all prime ideals in  $R$  implies that there is a fixed point in  $\mathbb{A}^n(k)$ .  $\square$

**Lemma 4.** *Let  $p$  be a prime and let  $\Gamma$  be a finite  $p$ -group. Let  $V$  be a connected differentiable manifold or an irreducible variety defined over a field  $k$  with  $p \neq \text{char}(k)$ , assume that  $\Gamma$  acts effectively on  $X$  and let  $v$  be a fixed point.*

Then  $\Gamma$  acts effectively on the (Zariski-) tangent space  $T_v V$ .

*Proof.* In the differentiable case finiteness of  $\Gamma$  permits us to construct an invariant Riemannian metric. Then  $\Gamma$  acts by isometries. Hence if  $\gamma \in \Gamma$  fixes a point  $v \in V$  and acts trivially on the tangent space, it preserves all the geodesics emanating from  $v$  and therefore induces the identity map on  $V$ .

In the algebraic case we note that, if  $\gamma \in \Gamma$  fixes  $v$  and acts trivially on the Zariski tangent space  $T_v V = (m_v/m_v^2)^*$ , then  $\gamma$  acts trivially on  $m_v^k/m_v^{k+1}$  for all  $k \in \mathbb{N}$ . Let  $f \in m_v$  with  $\gamma^*(f) \neq f$ . Then  $\gamma^*(f) - f \in m_v^k \setminus m_v^{k+1}$  for some  $k \in \mathbb{N}$ . But this implies that  $(\gamma^*)^n(f) - f \equiv n(\gamma^*f - f)$  modulo  $m_v^{k+1}$ . This contradicts our assumption that  $\gamma$  is of finite order coprime to the characteristic of  $k$ . Hence the assertion.  $\square$

Now we are in a position to prove the theorem.

*Proof.* If there would be such an injective group homomorphism, we would obtain an effective  $G$ -action on  $\mathbb{R}^n$  resp.  $\mathbb{A}^n$ .

Fix a prime  $p$ ,  $p \neq \text{char}(k)$ ,  $\text{char}(K)$ , let  $A = \mathbb{C}^* \times \mathbb{C}^*$  resp.  $A = k^* \times k^*$  and let  $\Gamma_r = A[p^r]$  denote the subgroup of torsion elements of order  $p^r$  in  $A$ . Let  $N_G(\Gamma_r)$  resp.  $Z_G(\Gamma_r)$  denote the normalizer resp. centralizer of  $\Gamma_r$  in  $G$ . Note that

$$\Gamma_r \simeq (\mathbb{Z}/p^r\mathbb{Z}) \times (\mathbb{Z}/p^r\mathbb{Z}), \quad N_G(\Gamma_r)/Z_G(\Gamma_r) \simeq \text{SL}_2(\mathbb{Z}/p^r\mathbb{Z})$$

and that  $N_G(\Gamma_r)/Z_G(\Gamma_r)$  acts effectively on  $\Gamma_r$  by group automorphisms.

Let  $V_r$  denote the fixed point set of the induced  $\Gamma_r$ -action on  $\mathbb{R}^n$  resp.  $\mathbb{A}^n$ . We claim that there exists some uniform (i.e. independent of  $r$ ) upper bound  $C < \infty$  for the number of (Zariski-)connected components of  $V_r$ . In the differentiable case it follows from Smith theory (see e.g. [12]) that  $V_r$  is connected. In the algebraic case the sequence  $V_r$  is a descending sequence of subvarieties of  $\mathbb{A}^n$  and hence there is some  $R$  such that  $V_r = V_R$  for all  $r \geq R$ . This yields the desired bound  $C$  for the number of Zariski connected components of  $V_r$ .

On the other hand, proposition 1 implies that none of the  $V_r$  can be empty. Lemma 4 implies that  $\Gamma_r$  acts effectively on  $T_x$  for every  $x \in V_r$ . The type of the representation of  $\Gamma_r$  on  $T_x$  is determined by the trace function  $\text{tr}_x$  defined by  $\gamma \mapsto \text{Tr}(\varrho_x(\gamma))$  where  $\varrho_x$  denotes the action of  $\Gamma_r$  on  $T_x$ . This trace function must depend continuously resp. regularly on  $x$ . However, there are only finitely many possible values. Hence, the trace function is locally constant with respect to  $x$ , i.e., on each (Zariski-) connected component of  $V_r$  all the representations  $\varrho_x$  of  $\Gamma_r$  are isomorphic.

For every number  $r \in \mathbb{N}$  fix a point  $x_r \in V_r$  and let  $N_r^0$  denote the subgroup of those elements of  $N_G(\Gamma_r)$  which stabilize the connected component of  $V_r$  containing  $x_r$ . For every  $\phi \in N_r^0$ ,  $\gamma \in \Gamma_r$ , the representations of  $\gamma^\phi = \phi\gamma\phi^{-1}$  and  $\gamma$  on  $T_{x_r}$  are isomorphic, because  $\phi(x_r)$  and  $x_r$  lie in the same connected component of  $V_r$ . It follows that the representations differ only by a permutation of the irreducible  $\Gamma_r$ -submodules of  $T_{x_r}$ . It follows that  $\# N_r^0/Z_G(\Gamma_r) \leq n!$ . Thus  $\# N(\Gamma_r)/Z(\Gamma_r) \leq C(n!)^2$ .

But  $\#N(\Gamma_r)/Z(\Gamma_r)$  goes to infinity for increasing  $r$ , because  $N(\Gamma_r)/Z(\Gamma_r) \simeq \text{SL}_2(\mathbb{Z}/p^r\mathbb{Z})$ . Thus we deduced a contradiction from the assumption that there exists such an injective group homomorphism.  $\square$

#### 4. The analytic case

Here and in the rest of our paper holomorphic embedding means a proper holomorphic embedding, i.e., a proper holomorphic map which is injective and immersive.

We will use the ideas developed in [16] together with the techniques for interpolation by automorphisms of  $\mathbb{C}^n$  developed in [3], [4], [9], [7], [10] to construct our more complicated embeddings for a complex subspace  $X$  of  $\mathbb{C}^n$  of any dimension.

If  $X$  is a Stein manifold of dimension  $n$ , then  $X$  can be embedded into  $\mathbb{C}^N$  with  $N = \max \left\{ 3, n + \left\lceil \frac{n}{2} \right\rceil + 1 \right\}$ . This bound is optimal, except possibly for  $n = 1$  (see [19]).

We want to point out that we give no results about existence of embeddings of a given Stein space into  $\mathbb{C}^n$ .

Instead we consider the question which further properties of such an embedding can be prescribed, if there already exists an embedding in the given dimension. Further results in this direction can be found in [7], [4], [5], [17].

**4.1. Results.** Let  $T$  be a topological space. By a continuous family of holomorphic selfmaps of a complex space  $X$  parameterized by  $T$  we mean a continuous map  $\alpha : T \times X \rightarrow X$  such that for every fixed  $t \in T$  the map  $\alpha(t, \cdot) : X \rightarrow X$  is holomorphic. If  $T$  is a real Lie group such that  $\alpha$  describes a group action, then the map  $\alpha$  is automatically real analytic (see for instance [2]). We will call a holomorphic map  $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$  nondegenerate if the Jacobian  $JF$  of  $F$  does not vanish identically. Our main result is the following theorem, which will be proved in the next paragraph.

**Theorem 4.** *Let  $X$  be a proper complex analytic subvariety of  $\mathbb{C}^n$  which consists of infinitely many points and  $\alpha : T \times X \rightarrow X$  a continuous family of holomorphic selfmaps of  $X$  parametrized by a locally compact topological space  $T$  with countable topology. Then there exists an embedding  $\varphi : X \hookrightarrow \mathbb{C}^n$  with the following property: If  $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$  is a nondegenerate holomorphic map such that*

1.  $F^{-1}(\mathbb{C}^n \setminus \varphi(X)) = \mathbb{C}^n \setminus \varphi(X)$  and
2.  $\varphi^{-1} \circ F \circ \varphi = \alpha(t, \cdot)$  for some  $t \in T$ ,

then  $F = \text{id}$ .

Theorem 3 stated in the introduction then arises as a corollary.

**Corollary 3.** *Let  $X$  be a complex space of dimension  $m$  which can be embedded into  $\mathbb{C}^n$  and such that the holomorphic automorphism group  $\text{Aut}_{\text{hol}}(X)$  is a Lie group. Also assume that  $m < n$  and  $X$  is not a finite set of points. Then there exists an embedding  $\varphi : X \hookrightarrow \mathbb{C}^n$  of  $X$  into  $\mathbb{C}^n$  such that the only holomorphic automorphism of  $\mathbb{C}^n$  leaving  $\varphi(X)$  invariant is the identity. If furthermore  $n - \dim(X) \geq 2$ , then  $\varphi$  can be chosen so that the holomorphic automorphism group  $\text{Aut}_{\text{hol}}(\mathbb{C}^n \setminus \varphi(X))$  of the complement is trivial.*

*Proof.* By definition, a Lie group is locally compact and of countable topology. Hence we can apply the theorem to  $X$  with  $T = \text{Aut}_{\text{hol}}(X)$ . Let  $\varphi : X \hookrightarrow \mathbb{C}^n$  be the embedding given by theorem 4. Any holomorphic automorphism  $\tilde{\alpha}$  of  $\mathbb{C}^n$  leaving  $\varphi(X)$  invariant is clearly a nondegenerate holomorphic map from  $\mathbb{C}^n$  to  $\mathbb{C}^n$  with  $\tilde{\alpha}^{-1}(\mathbb{C}^n \setminus \varphi(X)) = \mathbb{C}^n \setminus \varphi(X)$  and the restriction of  $\tilde{\alpha}$  to  $\varphi(X)$  is a holomorphic automorphism of  $X$ , hence an element of our family. This implies that  $\tilde{\alpha}$  is the identity map.

For the last statement of the corollary observe that, since  $\varphi(X)$  has at least codimension 2 in  $\mathbb{C}^n$ , any holomorphic automorphism  $\alpha \in \text{Aut}_{\text{hol}}(\mathbb{C}^n \setminus \varphi(X))$  of the complement extends to a holomorphic automorphism  $\tilde{\alpha}$  of  $\mathbb{C}^n$  and which leaves  $\varphi(X)$  invariant.  $\square$

**Corollary 4.** *Let  $X$  be a Stein manifold and let  $G$  be a Lie group acting effectively and by biholomorphic transformations on  $X$ .*

*Then there exists a natural number  $N$  and an embedding  $X \hookrightarrow \mathbb{C}^N$  such that for no  $g \in G \setminus \{e\}$  the induced automorphism of  $X$  can be extended to an automorphism of  $\mathbb{C}^N$ .*

If  $k = 1$ , then  $\text{Aut}_{\text{hol}}(\mathbb{C}^k)$  is a Lie group. Hence we obtain the following corollary.

**Corollary 5.** *For every  $n \geq 2$  there exists an embedding  $\varphi : \mathbb{C} \hookrightarrow \mathbb{C}^n$  such that the only holomorphic automorphism of  $\mathbb{C}^n$  leaving  $\varphi(\mathbb{C})$  invariant is the identity. If  $n \geq 3$ , then  $\varphi$  can be chosen such that  $\text{Aut}_{\text{hol}}(\mathbb{C}^n \setminus \varphi(\mathbb{C}))$  is trivial.*

**Question 1.** Does there exist an embedding  $\varphi : \mathbb{C}^k \rightarrow \mathbb{C}^n$  with

$$\text{Aut}_{\text{hol}}(\mathbb{C}^n \setminus \varphi(\mathbb{C}^k)) = \{\text{id}\}$$

for every  $0 < k < n$ ?

**Proposition 2.** *Let  $X \subset \mathbb{C}^n$  be a complex analytic hypersurface.*

*Then there exists an embedding  $\varphi : X \rightarrow \mathbb{C}^n$  such that  $\text{Aut}_{\text{hol}}(\mathbb{C}^n \setminus X)$  is countable.*

*Proof.* In [4] Buzzard and Fornæss proved (see also [7], Theorem 5.1 for a more general statement) that there is an embedding  $\varphi : X \hookrightarrow \mathbb{C}^n$  such that the complement  $\mathbb{C}^n \setminus \varphi(X)$  is Kobayashi-hyperbolic. Hence the result follows from the lemma below.  $\square$

**Lemma 5.** *Let  $V$  be an affine algebraic manifold (e.g.  $V = \mathbb{C}^n$ ) and  $Y \subset V$  be a complex analytic subvariety such that the complement  $V \setminus Y$  is Kobayashi-hyperbolic. Then  $\text{Aut}_{\text{hol}}(V \setminus Y)$  is countable and discrete in the compact-open topology.*

*Proof.* The group of holomorphic automorphisms of a Kobayashi-hyperbolic manifold is a closed subgroup of the isometry group with respect to the Kobayashi distance and is isomorphic to a real Lie group. We must show that the identity component of  $\text{Aut}_{\text{hol}}(V \setminus Y)$  is zero-dimensional. If  $\text{Aut}_{\text{hol}}(V \setminus Y)$  were positive dimensional, then any nonzero vector  $v$  in the Lie algebra would give rise to a non-trivial action of the additive group  $(\mathbb{R}, +)$  on  $V \setminus Y$  given by the map  $(t, z) \mapsto \exp(tv) \cdot z$ . Since  $V \setminus Y$  is Stein and has no non-constant bounded plurisubharmonic functions we can apply Cor. 2.2. in [8] to conclude that our action of  $(\mathbb{R}, +)$  extends to an  $(\mathbb{C}, +)$  action on  $V \setminus Y$ . This contradicts the fact that every holomorphic map from  $\mathbb{C}$  into the Kobayashi-hyperbolic manifold  $V \setminus Y$  is constant.  $\square$

**Question 2.** Given a complex analytic hypersurface  $X \subset \mathbb{C}^n$ , does there exist an embedding  $j: X \hookrightarrow \mathbb{C}^n$  such that the automorphism group of  $\mathbb{C}^n \setminus j(X)$  is trivial?

More generally, one may pose the following question.

**Question 3.** Given a complex analytic hypersurface  $X \subset \mathbb{C}^n$  and a connected complex Lie group  $G$ , under which conditions is it true that there exists an embedding  $j: X \hookrightarrow \mathbb{C}^n$  such that  $\text{Aut}_{\text{hol}}(\mathbb{C}^n \setminus X) = G$ ?

One may also ask the same question for  $\{\phi \in \text{Aut}_{\text{hol}}(\mathbb{C}^n) : \phi(j(X)) = j(X)\}$  instead of  $\text{Aut}_{\text{hol}}(\mathbb{C}^n \setminus (j(X)))$ .

**4.2. Proofs.** As already mentioned above, in the proof of theorem 4 we will use the techniques developed in [7], [16] (see also [9], [4]). For the convenience of the reader we state here all results from these papers which will be used in the sequel.

**Proposition 3** (see [7], Prop. 1.1). *Let  $n > 1$ . Assume that*

- (a)  $K \subset \mathbb{C}^n$  is a compact polynomially convex set,
- (b)  $\{a_j\}_{j=1}^s \subset K$  is a finite subset of  $K$ ,
- (c)  $p$  and  $q$  are arbitrary points in  $\mathbb{C}^n \setminus K$  (not necessarily distinct),
- (d)  $N$  is a nonnegative integer, and
- (e)  $P: \mathbb{C}^n \rightarrow \mathbb{C}^n$  is a holomorphic polynomial map of degree at most  $m \geq 1$  with  $P(0) = 0$  and  $JP(0) \neq 0$ .

*Then for each  $\varepsilon > 0$  there exists an automorphism  $F \in \text{Aut}_{\text{hol}}(\mathbb{C}^n)$  satisfying*

- (i)  $F(p) = q$  and  $F(z) = q + P(z - p) + O(|z - p|^{m+1})$  as  $z \rightarrow p$ ,
- (ii)  $F(z) = z + O(|z - a_j|^n)$  as  $z \rightarrow a_j$  for each  $j = 1, 2, \dots, s$  and
- (iii)  $|F(z) - z| + |F^{-1}(z) - z| < \varepsilon \quad \forall z \in K$ .

**Proposition 4** (see [7], Prop. 4.1 and 4.2). *Let  $K_0 \subset K_1 \subset K_2 \subset \dots \subset \bigcup_{j=0}^{\infty} K_j = \mathbb{C}^n$  be compact sets such that  $K_{j-1} \subset \text{Int } K_j$  for each  $j \in \mathbb{N}$ . Suppose  $\varepsilon_j$  ( $j = 1, 2, 3, \dots$ ) are real numbers such that*

$$(1) \quad 0 < \varepsilon_j < \text{dist}(K_{j-1}, \mathbb{C}^n \setminus K_j) \quad (j \in \mathbb{N}), \quad \sum_{j=1}^{\infty} \varepsilon_j < \infty.$$

*Suppose that for each  $j = 1, 2, 3, \dots$   $\Psi_j$  is a holomorphic automorphism of  $\mathbb{C}^n$  satisfying*

$$(2) \quad |\Psi_j(z) - z| < \varepsilon_j, \quad z \in K_j.$$

*Set  $\Phi_m = \Psi_m \circ \Psi_{m-1} \circ \dots \circ \Psi_1$ . Then there is an open set  $\Omega \subset \mathbb{C}^n$  such that  $\lim_{m \rightarrow \infty} \Phi_m = \Phi$  exists on  $\Omega$  (uniformly on compacts), and  $\Phi$  is a biholomorphic map of  $\Omega$  onto  $\mathbb{C}^n$ . In fact,  $\Omega = \bigcup_{j=1}^{\infty} \Phi_m^{-1}(K_m)$ . Also  $\Omega$  can be characterized as the set of points  $z \in \mathbb{C}^n$  such that the sequence  $\{\Phi_m(z) : m \in \mathbb{N}\}$  is bounded.*

As usual  $B_r$  denotes the (open) ball of radius  $r > 0$  in  $\mathbb{C}^n$  and  $\partial B_r$  denotes its boundary.

**Lemma 6** (see [16], Lemma 4.3). *Let  $0 < a_1 < a_2, 0 < r_1 < r_2, c > 0$  be real numbers and  $A \subset \partial B_{r_1}$  a dense subset. Then there exists a finite subset  $\bigcup_{i=1}^m z_i$  of  $A$  such that*

$$F(B_{a_1}) \subset B_{r_1}$$

*for all holomorphic maps  $F: B_{a_2} \rightarrow B_{r_2}$  with  $F(0) \in B_{\frac{r_1}{2}}, |JF(0)| \geq c$  and  $z_i \notin F(B_{a_2}) \forall i = 1, 2, \dots, m$ .*

**Remark.** Lemma 1 is proved in [16] with  $A = \partial B_{a_1}$ . The proof starts with an arbitrary countable dense subset  $x_1, x_2, \dots$  of  $\partial B_{a_1}$  and the desired finite set is constructed as a subset of  $x_1, x_2, \dots$ . So the only minor modification to be made in the proof is to start with  $x_1, x_2, \dots$  being a subset of  $A$  which is dense in  $\partial B_{a_1}$ .

The proof of theorem 4 consists of two steps. The first step is to construct an embedding in such a way that conditions 1. and 2. of the theorem force a nondegenerate holomorphic map  $F: \mathbb{C}^n \rightarrow \mathbb{C}^n$  to be affine, i.e., an affine automorphism of  $\mathbb{C}^n$ . The second point is to ensure that the only affine automorphism of  $\mathbb{C}^n$  leaving the embedded variety  $X$  invariant is the identity. To be accurate in the second point we need one more technical result to be explained now:

By a submanifold  $Z \subset \mathbb{C}^n$  we shall mean an injectively immersed holomorphic submanifold (not necessarily closed).

**Definition 1.** Let  $k \geq 2$  be a natural number. We say that a submanifold  $X$  of  $\mathbb{C}^n$  osculates of order  $k$  at some point  $x \in X$  if  $X$  has contact order  $k$  with the tangent space  $T_x X \subset \mathbb{C}^n$  at  $x$ .

In local coordinates this condition can be expressed in the following way:

Let  $\varphi : U(\subset \mathbb{C}^m) \rightarrow X$  be a holomorphic coordinate chart for the  $m$ -dimensional manifold  $X$  around  $x$  (i.e.  $\varphi(0) = x$ ). Then  $X$  osculates of order  $k$  at  $x$  if and only if

$$\frac{\partial}{\partial w^\alpha} \Big|_{w=0} \varphi \in T_x X \text{ for every multiindex } \alpha = (\alpha_1, \alpha_2, \dots, \alpha_m) \text{ with } 2 \leq |\alpha| \leq k.$$

The property to osculate of order  $k$  is evidently preserved by affine coordinate changes on  $\mathbb{C}^n$ , i.e., if  $\psi : \mathbb{C}^n \rightarrow \mathbb{C}^n$  is an affine automorphism of  $\mathbb{C}^n$ , then a submanifold  $X \subset \mathbb{C}^n$  osculates of order  $k$  at  $x \in X$  iff the submanifold  $\psi(X)$  osculates of  $k$  at  $\psi(x) \in \psi(X)$ .

**Lemma 7.** *Let  $M \subset \mathbb{C}^n$  be an  $m$ -dimensional ( $m < n$ ) submanifold. Suppose we are given*

- (a) *a compact subset  $K_M$  of  $M$ ,*
- (b) *a compact subset  $K$  of  $\mathbb{C}^n$ ,*
- (c) *finitely many points  $a_1, a_2, \dots, a_r$  in  $K_M \subset M$ ,*
- (d) *finitely many points  $b_1, b_2, \dots, b_q$  in  $M \setminus K_M$ ,*
- (e) *a natural number  $k \geq 2$ , if  $m = 1$  and  $n = 2$  then  $k \geq 3$ , and*
- (f) *a real number  $\varepsilon > 0$ .*

*Then there exists a holomorphic automorphism  $\psi \in \text{Aut}_{\text{hol}}(\mathbb{C}^n)$  with the following properties:*

- 1.  $\psi(a_i) = a_i \quad \forall i = 1, 2, \dots, r,$
- 2.  $\psi(z) = z + O(|z - b_i|^{k+1})$  as  $z \rightarrow b_i \quad \forall i = 1, 2, \dots, q,$
- 3.  $|\psi(z) - z| + |\psi^{-1}(z) - z| < \varepsilon \quad \forall z \in K,$
- 4. *there is no point  $m \in K_M$  such that the submanifold  $\psi(M)$  of  $\mathbb{C}^n$  osculates of order  $k$  at  $\psi(m)$ .*

Before we can prove lemma 7 we need a sublemma. The notations used in the sublemma are the same as those in lemma 7.

**Sublemma.** *For each point  $p \in K_M$  there is an open neighborhood  $U_p$  of  $p$  in  $M$  and a family  $\psi_t$  of automorphisms of  $\mathbb{C}^n$  parametrized by  $\mathbb{C}^{N(m,n,k)}$  with*

$$N(m, n, k) = \left[ \binom{m+k}{m} - (m+1) \right] \cdot (n-m)$$

*such that:*

- 1.  $\psi_0 = \text{id}.$

2. Every  $\psi_t$  fulfills conditions 1. and 2. of lemma 7.
3. There exists an open neighbourhood  $T$  of 0 in  $\mathbb{C}^{N(m,n,k)}$  such that

$$\Sigma = \{t \in T : \exists p' \in U_p : \psi_t(M) \text{ of } \mathbb{C}^n \text{ osculates of order } k \text{ at } \psi_t(p')\}$$

is a set of Lebesgue measure zero.

*Proof.* If  $M$  does not osculate of order  $k$  at  $p$ , then  $\psi_t = \text{id}$  for all  $t$  does the job. Hence we may assume that  $M$  osculates of order  $k$  at  $p$ .

Without loss of generality we may assume that  $p = 0$ ,  $T_p M = \{(z_1, \dots, z_m, 0, \dots, 0)\}$ . Let  $\pi : \mathbb{C}^n \rightarrow \mathbb{C}^m$  denote the map given by projection onto the first  $m$  coordinates. After a linear change of coordinates we may assume that  $\pi(b_i) \neq 0 \in \mathbb{C}^m$  for all  $1 \leq i \leq q$ .

To examine whether  $\psi(M)$  osculates of order  $k$  at some point  $\psi(p')$  with  $p' \in M$  for a given  $\psi \in \text{Aut}_{\text{hol}}$  we consider the map  $F^\psi : M \rightarrow \mathbb{C}^{N(m,n,k)}$  whose coordinate functions  $F_{\alpha,u}^\psi$  are enumerated by pairs  $(\alpha, u)$  where  $\alpha$  is a multiindex  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m)$  with  $2 \leq |\alpha| \leq k$  and  $u \in \mathbb{N}$  satisfies  $m + 1 \leq u \leq n$ , these are given by

$$F_{\alpha,u}^\psi(w) = \det \begin{pmatrix} \frac{\partial}{\partial w_1} (\psi)_1(w) & \cdots & \frac{\partial}{\partial w_1} (\psi)_m(w) & \frac{\partial}{\partial w_1} (\psi)_u(w) \\ \vdots & & \vdots & \vdots \\ \frac{\partial}{\partial w_m} (\psi)_1(w) & \cdots & \frac{\partial}{\partial w_m} (\psi)_m(w) & \frac{\partial}{\partial w_m} (\psi)_u(w) \\ \frac{\partial}{\partial w^\alpha} (\psi)_1(w) & \cdots & \frac{\partial}{\partial w^\alpha} (\psi)_m(w) & \frac{\partial}{\partial w^\alpha} (\psi)_u(w) \end{pmatrix}.$$

Here  $(\psi)_i$  denotes the  $i$ -th coordinate function of the map  $\psi : \mathbb{C}^n \rightarrow \mathbb{C}^n$  and  $(w_i)_{1 \leq i \leq m}$  is some fixed system of local coordinates on  $M$  near  $p$ . Then  $\psi(M)$  osculates of order  $k$  at  $w$  if and only if  $F^\psi(w) = 0$ .

By restricting our attention to a small enough neighbourhood of  $p \in M$  we may choose  $w_i = z_i$  ( $1 \leq i \leq m$ ).

Now we come to the explicit construction of our family  $(\psi_t)_t$  of automorphisms of  $\mathbb{C}^n$ .

For each pair  $(\alpha, u)$  with  $\alpha$  and  $u$  as above we choose a holomorphic function  $h_{\alpha,u}$  on  $\mathbb{C}^m$  such that

1.  $h_{\alpha,u} - z^\alpha$  vanishes of order at least  $k + 1$  in 0,
2.  $h_{\alpha,u}$  vanishes of order at least  $k + 1$  for all  $\pi(b_i)$  ( $1 \leq i \leq q$ ),
3.  $h_{\alpha,u}$  vanishes at all  $\pi(a_j)$  ( $1 \leq j \leq r$ ).

Next we define a map  $\psi : \mathbb{C}^{N(m,n,k)} \times \mathbb{C}^n \rightarrow \mathbb{C}^n$  by

$$\psi(t, z) = z + \sum_{(\alpha, u)} t_{(\alpha, u)} h_{\alpha, u}(z_1, \dots, z_m) e_u.$$

Here  $e_u$  denotes the  $u$ -th unit vector and the coordinates of  $\mathbb{C}^{N(m, n, k)}$  are indexed by pairs  $(\alpha, u)$  as above. For every  $t \in \mathbb{C}^{N(m, n, k)}$  the map  $\psi_t = \psi(t, \cdot)$  is an automorphism of  $\mathbb{C}^n$  fulfilling (because of 2. and 3.) conditions 1. and 2. of lemma 7. Furthermore  $\psi_0 = \text{id}$ .

Easy calculations (using 1.) show that

$$\frac{\partial}{\partial t_{(\alpha, u)}} F_{\alpha, u}^\psi|_{w=0} = \alpha!$$

and

$$\frac{\partial}{\partial t_{(\alpha', u')}} F_{\alpha', u'}^\psi|_{w=0} = 0$$

whenever  $u \neq u'$  or whenever  $u = u'$ ,  $|\alpha'| \leq |\alpha|$  and  $\alpha' \neq \alpha$ .

This implies that the map  $\Phi : \mathbb{C}^{N(m, n, k)} \times M \rightarrow \mathbb{C}^{N(m, n, k)}$  defined by  $\Phi(t, z) = F^{\psi_t}(z)$  has maximal rank near 0. Thus there exists an open neighborhood  $\Omega_p$  of the form  $\Omega_p = T \times U_p$  of  $(0, p)$  in  $\mathbb{C}^{N(m, n, k)} \times M$  such that  $\Phi|_{\Omega_p}$  is transversal to  $0 \in \mathbb{C}^{N(m, n, k)}$ . This implies that for almost all  $t \in T$  the map  $F^{\psi_t} : U_p \rightarrow \mathbb{C}^{N(m, n, k)}$  is transversal to 0. Since  $m < N(m, n, k)$  (here we need  $k > 2$  if  $n = 2$  and  $m = 1$ , in all other cases  $k = 2$  is already sufficient) this means that for almost all  $t \in T$  the image  $F^{\psi_t}(U_p)$  does not meet 0, i.e.  $\psi_t(M)$  does not osculate of order  $k$  for any  $p' \in U_p$ .  $\square$

*Proof of lemma 7.* Choose finitely many open subsets  $U_i$  of  $M$  together with families  $\psi^i : T_i \times \mathbb{C}^n \rightarrow \mathbb{C}^n$  of automorphisms  $i = 1, 2, \dots, l$  as in the sublemma and choose compact subsets  $K_i \subset U_i$  of the  $U_i$  which cover  $K_M$ . Since  $\psi_0^1$  is the identity, for  $t$  sufficiently small the automorphism  $\psi_t^1$  moves no point of  $K_M$  more than  $\frac{\varepsilon}{l}$ . So we find a  $t_1 \in T_1$  such that  $|\psi_{t_1}^1(z) - z| < \frac{\varepsilon}{l} \forall z \in K$  and the submanifold  $\psi_{t_1}^1(M)$  does not osculate of order  $k$  at any point of  $\psi_{t_1}^1(K_1)$ .

Observe that the property of not osculating of order  $k$  at some point is preserved under small perturbations, i.e., for each compact subset  $L$  of a submanifold  $M$  of  $\mathbb{C}^n$  which does not osculate of order  $k$  at any point of  $L$  there exists some  $\varepsilon$  such that for each automorphism  $\Psi$  of  $\mathbb{C}^n$  the property  $|\Psi(z) - z| < \varepsilon \forall z \in L$  implies that  $\Psi(M)$  remains non-osculating of order  $k$  at any point of  $\Psi(L)$  (for holomorphic maps small perturbations in values imply small perturbations in derivatives). Hence we find a sufficiently small  $t_2 \in T_2$  such that first  $|\psi_{t_2}^2(z) - z| < \frac{\varepsilon}{l} \forall z \in \psi^1(K)$ , second the submanifold  $\psi_{t_2}^2 \circ \psi_{t_1}^1(M)$  does not osculate of order  $k$  at any point of  $\psi_{t_2}^2 \circ \psi_{t_1}^1(K_2)$  and third  $\psi_{t_2}^2 \circ \psi_{t_1}^1(M)$  remains non osculating of order  $k$  at any point of  $\psi_{t_2}^2 \circ \psi_{t_1}^1(K_1)$ . Proceeding by induction we find an automorphism  $\psi := \psi_{t_l}^l \circ \psi_{t_{l-1}}^{l-1} \circ \dots \circ \psi_{t_1}^1$  moving no point of  $K$  more than  $\varepsilon$  and such that  $\psi(M)$  does not osculate of order  $k$  at any point of  $\psi \left( \bigcup_{i=1}^l K_i \right) \supset \psi(K_M)$ . Since all automorphisms  $\psi^i$  satisfy properties 1. and 2.,  $\psi$  satisfies them as well.  $\square$

*Proof of theorem 3.* In the case that  $X$  is a countable set of points the theorem is proved in [16] where these sets are called rigid. So we will deal only with the case where  $X$  is of positive dimension.

Let  $\varrho : X \rightarrow \mathbb{R}^{\geq 0}$  be a continuous exhaustion function, i.e.,  $X_r := \varrho^{-1}([0, r])$  is a compact subset of  $X$  for all  $r \geq 0$ . Also let us denote by  $\tilde{X}$  the union of all components of the smooth part of  $X$  which have maximal dimension, say  $m$ ,  $0 < m < n$ . We fix a natural number  $k \geq 2$ . If  $n = 2$  and  $m = 1$  we require  $k \geq 3$ . Let  $b_1, b_2, \dots, b_{(n+2)} \in B_1 \subset \mathbb{C}^n$  be points such that no affine automorphism of  $\mathbb{C}^n$  except the identity can permute these points. Choose  $x_1, x_2, \dots, x_{n+2} \in \tilde{X}$  to be mutually distinct points. Applying Prop. 3 several times (see also [7], Corollary 1.2), we find an automorphism  $\Psi \in \text{Aut}_{\text{hol}}(\mathbb{C}^n)$  which maps  $x_i$  to  $b_i$  with the property that the submanifold  $\Psi(\tilde{X})$  of  $\mathbb{C}^n$  osculates of order  $k$  at the points  $\Psi(x_i) = b_i$  for each  $i = 1, 2, \dots, n + 2$ . We denote the embedding  $\Psi \circ i : X \hookrightarrow \mathbb{C}^n$  by  $\varphi_0$ .

Our aim is to construct a Fatou-Bieberbach domain  $\Omega \subset \mathbb{C}^n$  containing  $\varphi_0(X)$  together with a biholomorphic map  $\Phi : \Omega \rightarrow \mathbb{C}^n$  onto  $\mathbb{C}^n$  such that the embedding  $\varphi := \Phi \circ \varphi_0$  has the desired property. The map  $\Phi$  will be constructed as a limit of automorphisms  $\Phi_m \in \text{Aut}_{\text{hol}}(\mathbb{C}^n)$ .

We start with an exhaustion  $\bigcup_{i=1}^{\infty} T_i = T$  of the topological space  $T$  by compact subsets  $T_i$  ( $T_i \subset \text{Int} T_{i+1} \subset T$ ). Also we choose a sequence of open relatively compact neighborhoods  $U_i$  ( $i = 1, 2, 3, \dots$ ) of the set  $\bigcup_{i=1}^{n+2} \{x_i\}$  in  $X$  with  $\bigcap_{i=1}^{\infty} U_i = \bigcup_{i=1}^{n+2} \{x_i\}$ .

We will now inductively define real numbers  $\varepsilon_m, R_m > 0$ , natural numbers  $k(m)$ , finite subsets  $\bigcup_{j=1}^{k(m)} \{a_j^m\}$  of  $\partial B_{m+1}$ , finite subsets  $\left\{ \bigcup_{j=1}^{k(m)} x_j^m \right\}$  of  $X$ , and automorphisms  $\Phi_m$  of  $\mathbb{C}^n$  for  $m = 0, 1, \dots$ . The beginning point is  $\varepsilon_0 = \frac{3}{4}$ ,  $R_0 = 1$ ,  $k(0) = 0$  and  $\Phi_0 = \text{id}$ . For  $m \geq 1$  these data are recursively constructed in such a way that the following conditions are fulfilled:

$$(1_m) \quad 0 < \varepsilon_m < \frac{\varepsilon_{m-1}}{3}.$$

$$(2_m) \quad \text{If } F : B_1 \rightarrow B_{m+2} \setminus \left\{ \bigcup_{j=1}^{k(m)} a_j^m \right\} \text{ is a holomorphic map with } \|F(0)\| \leq \frac{m+1}{2} \text{ and } JF(0) \geq 1 \text{ then } F(B_{1-\frac{\varepsilon_m}{2}}) \subset B_{m+1}.$$

$$(3_m) \quad \Phi_m \circ \varphi_0(x_j^m) = a_j^m \text{ and } \varrho(x_j^m) > \max_{x \in (\Phi_{m-1} \circ \varphi_0)^{-1}(\bar{B}_m), t \in T_m} \varrho(\alpha(t, x)).$$

$$(4_m) \quad \|\Phi_m \circ \varphi_0(x) - \Phi_{m-1} \circ \varphi_0(x)\| \leq \varepsilon_m \text{ for all } x \in X_{R_{m-1}}.$$

$$(5_m) \quad \|\Phi_m \circ \Phi_{m-1}^{-1}(z) - z\| \leq \varepsilon_m \text{ for } z \in \bar{B}_m.$$

$$(6_m) \quad \Phi_m \circ \varphi_0(x_j^l) = a_j^l \quad \forall l < m, j = 1, 2, \dots, k(l).$$

$$(7_m) \quad \Phi_m \circ \Phi_{m-1}^{-1}(z) = z + O(|z - b_i|^{k+1}) \text{ as } z \rightarrow b_i \text{ for each } i = 1, 2, \dots, n + 2.$$

(8<sub>m</sub>) The submanifold  $\Phi_m \circ \varphi_0(\tilde{X})$  of  $\mathbb{C}^n$  does not osculate of order  $k$  at any point  $\Phi_m \circ \varphi_0(x)$  with  $x \in (X_{R_{m-1}} \cup \tilde{X}) \setminus U_m$ .

$$(9_m) \quad \|\Phi_m \circ \varphi_0(x)\| \geq m + 1 \quad \forall x \in X \setminus X_{R_m}.$$

$$(10_m) \quad R_m > R_{m-1} + 1.$$

We will now confirm that such a recursive construction is possible. For step 1 of the induction we first choose  $\varepsilon_1 < \frac{\varepsilon_0}{3}$  and, using lemma 1, let  $\bigcup_{j=1}^{k(1)} \{a_j^1\}$  be a finite subset of  $\partial B_2 \setminus \varphi_0(X)$  fulfilling (2<sub>1</sub>) (for the set  $A$  in lemma 1 take  $\partial B_2 \setminus \varphi_0(X)$ ). Consider the compact set  $K := \bar{B}_1 \cup \varphi_0(X_{R_0})$ . By Lemma 7 in [9] the polynomially convex hull  $\hat{K}$  of  $K$  is contained in  $\bar{B}_1 \cup \varphi_0(X)$ ; in particular it does not contain any of the points  $a_j^1$ . We can choose distinct points  $x_1^1, x_2^1, \dots, x_{k(1)}^1$  in  $X \setminus (\hat{K} \cup X_{R_0})$  such that

$$\varrho(x_i^1) > \max_{x \in \varphi_0^{-1}(B_1)} \varrho(x), \quad i = 1, 2, \dots, k(1)$$

and use  $k(1)$  times proposition 3 to find an automorphism  $\Phi_1^1$  with  $\|\Phi_1^1(z) - z\| \leq \frac{\varepsilon_1}{2}$  for  $z \in K$ ,  $\Phi_1^1 \circ \varphi_0(x_j^1) = a_j^1, j = 1, 2, \dots, k(1)$  and  $\Phi_1^1(z) = z + O(|z - b_i|^{k+1})$  as  $z \rightarrow b_i, i = 1, 2, \dots, n + 2$ . By lemma 7 we find another automorphism  $\Phi_1^2$  which moves the compact set  $\Phi_1^1(K)$  less than  $\frac{\varepsilon_1}{2}$ , fixes the points  $a_j^1, j = 1, 2, \dots, k(1)$ , matches the identity up to order  $k$  at the points  $b_i, i = 1, 2, \dots, n + 2$  and has the property that the submanifold  $\Phi_1^2 \circ \Phi_1^1 \circ \varphi_0(\tilde{X})$  does not osculate of order  $k$  at any point  $\Phi_1^2 \circ \Phi_1^1 \circ \varphi_0(x)$  with  $x \in (X_{R_0} \cup \tilde{X}) \setminus U_1$ . The composition  $\Phi_1 := \Phi_1^2 \circ \Phi_1^1$  (together with the set  $\bigcup_{i=1}^{k(1)} \{x_i^1\}$ ) satisfies all properties from (3<sub>1</sub>) to (8<sub>1</sub>). Finally we choose  $R_1$  big enough to satisfy (9<sub>1</sub>).

The description of the  $m$ -th step is similar to the first step. To be accurate we carry it out in detail. Suppose we have already constructed  $\varepsilon_i, R_i > 0$ , finite subsets  $\bigcup_{j=1}^{k(i)} \{a_j^i\}$  of  $\partial B_{i+1}$ , finite subsets  $\bigcup_{j=1}^{k(i)} \{x_j^i\}$  of  $X$  together with automorphisms  $\Phi_i$  satisfying (1<sub>i</sub>) up to (9<sub>i</sub>) for all  $i$  from 1 to  $m - 1$ . Again we first choose  $\varepsilon_m \leq \varepsilon_{m-1}$  so small that any perturbation of the embedding  $\Phi_{m-1} \circ \varphi_0 : \tilde{X} \hookrightarrow \mathbb{C}^n$  which is smaller than  $3\varepsilon_m$  on the compact set  $(X_{R_{m-2}} \cup \tilde{X}) \setminus U_{m-1}$  does not destroy the property that  $\Phi_{m-1} \circ \varphi_0(\tilde{X})$  does not osculate of order  $k$  at any point in the image of  $(X_{R_{m-2}} \cup \tilde{X}) \setminus U_{m-1}$ . Next, using lemma 1, we choose a finite subset  $\bigcup_{j=1}^{k(m)} \{a_j^m\}$  of  $\partial B_{m+1} \setminus \Phi_{m-1} \circ \varphi_0(X)$  fulfilling (2<sub>m</sub>). Consider the compact set  $K := \bar{B}_m \cup \Phi_{m-1} \circ \varphi_0(X_{R_{m-1}})$ . Again by lemma 7 in [9] the polynomially convex hull  $\hat{K}$  of  $K$  is contained in  $\bar{B}_m \cup \Phi_{m-1} \circ \varphi_0(X)$ . Hence it does not contain any of the points  $a_j^m$ . We choose distinct points  $x_1^m, x_2^m, \dots, x_{k(m)}^m$  in  $X \setminus (\hat{K} \cup X_{R_{m-1}})$  such that

$$\varrho(x_i^m) > \max_{x \in \Phi_{m-1} \circ \varphi_0^{-1}(B_m)} \varrho(x), \quad i = 1, 2, \dots, k(m)$$

and use  $k(m)$  times proposition 3 to find an automorphism  $\Psi_m^1$  with  $\|\Psi_m^1(z) - z\| \leq \frac{\varepsilon_m}{2}$  for  $z \in K$ ,  $\Psi_m^1 \circ \varphi_0(x_j^m) = a_j^m$ ,  $j = 1, 2, \dots, k(m)$ ,  $\Psi_m^1(a_j^l) = a_j^l$ ,  $j = 1, 2, \dots, k(l) \forall l \leq m-1$ , and  $\Psi_m^1(z) = z + O(|z - b_i|^{k+1})$  as  $z \rightarrow b_i$ ,  $i = 1, 2, \dots, n+2$ . By lemma 7 we find another automorphism  $\Psi_m^2$  which moves the compact set  $\Psi_m^1(K)$  less than  $\frac{\varepsilon_m}{2}$ , fixes the points  $a_j^l$ ,  $j = 1, 2, \dots, k(l)$ ,  $1 \leq l \leq m$ , matches the identity up to order  $k$  at the points  $b_i$ ,  $i = 1, 2, \dots, n+2$ , and has the property that the submanifold  $\Psi_m^2 \circ \Psi_m^1 \circ \varphi_0(\tilde{X})$  does not osculate of order  $k$  at any point in the image of  $(X_{R_{m-1}} \cup \tilde{X}) \setminus U_m$ . We set  $\Psi_m := \Psi_m^2 \circ \Psi_m^1$  and  $\Phi_m = \Psi_m \circ \Phi_{m-1}$ . Now all conditions  $(3_m)$  to  $(8_m)$  are also satisfied. Finally we choose  $R_m$  big enough to satisfy  $(9_m)$  and  $(10_m)$ .

By proposition 2, the properties  $(5_m)$  together with the fact that  $\varepsilon_m < \frac{1}{3^m}$  imply that  $\lim_{m \rightarrow \infty} \Phi_m$  exists (uniformly on compacts) on  $\Omega := \bigcup_{m=1}^{\infty} \Phi_m^{-1}(B_m)$  and defines a biholomorphic map from  $\Omega$  onto  $\mathbb{C}^n$ .

If  $z \in \varphi_0(X)$ , then we find an  $m$  such that  $x := \varphi_0^{-1} \in X_{R_m}$ . So properties  $(4_m)$  imply:

$$\|\Phi_l \circ \varphi_0(x) - \Phi_m \circ \varphi_0(x)\| \leq \sum_{k=m}^{l-1} \|\Phi_{k+1} \circ \varphi_0(x) - \Phi_k \circ \varphi_0(x)\| \leq \sum_{k=m}^{l-1} \varepsilon_{k+1} < \sum_{i=1}^{\infty} \varepsilon_i < \infty.$$

This shows that the set  $\{\Phi_m(z) = \Phi_m \circ \varphi_0(x) : m \in \mathbb{N}\}$  is bounded in  $\mathbb{C}^n$ . By proposition 2 this means  $z \in \Omega$ . We have proved  $\varphi_0(X) \subset \Omega$ . This shows that  $\varphi := \Phi \circ \varphi_0$  is a (proper holomorphic) embedding  $\varphi : X \hookrightarrow \mathbb{C}^n$ .

We now show that  $\varphi$  satisfies the conclusion of the theorem. First observe that  $(6_m)$  for all  $m$  implies

$$\varphi(x_j^m) = a_j^m, \quad j = 1, 2, \dots, k(m) \quad \forall m \in \mathbb{N}.$$

We set  $\alpha = \sum_{i=1}^{\infty} \varepsilon_i$ . By  $(1_m)$  we have  $\alpha < \frac{1}{2}$ . Next we show:

$$(*) \quad \varphi^{-1}(B_{m-1}) \subset (\Phi_m \circ \varphi_0)^{-1}(B_m) \quad \forall m \in \mathbb{N}.$$

Let  $x \in \varphi^{-1}(B_{m-1})$  be an arbitrary point. Since this means  $\Phi \circ \varphi_0(x) \in B_{m-1}$ , we may choose  $k_0 > m$  and  $\varepsilon = 1 - 2\alpha > 0$ , such that

$$(**) \quad \Phi_k \circ \varphi_0(x) \in B_{m-1+\varepsilon} \quad \forall k \geq k_0.$$

By  $(5_{m+1})$  and Rouché's theorem (see [6], p.110) we have  $\Psi_{m+1}(B_m) \supset B_{m-2\varepsilon_m}$ , i.e.,  $B_m \supset \Psi_{m+1}^{-1}(B_{m-2\varepsilon_m})$ . Hence  $\Phi_m^{-1}(B_m) \supset \Phi_m^{-1}(\Psi_{m+1}^{-1}(B_{m-2\varepsilon_m})) = \Phi_{m+1}^{-1}(B_{m-2\varepsilon_m})$ . By induction, using  $(5_{m+2}), \dots, (5_k)$  we find

$$\Phi_m^{-1}(B_m) \supset \Phi_k^{-1}(B_{m-2\sum_{i=m}^{k-1} \varepsilon_i}) \supset \Phi_k^{-1}(B_{m-2\alpha}).$$

Together with (\*\*) this implies by our choice of  $\varepsilon$  that  $\varphi_0(x) \in \Phi_m^{-1}(B_m)$  and hence  $x \in (\Phi_m \circ \varphi_0)^{-1}(B_m)$ . So (\*) is proved.

Now suppose  $F: \mathbb{C}^n \rightarrow \mathbb{C}^n$  is a nondegenerate holomorphic map with

$$F^{-1}(\mathbb{C}^n \setminus \varphi(X)) = \mathbb{C}^n \setminus \varphi(X) \quad \text{and} \quad \varphi^{-1} \circ F \circ \varphi: X \rightarrow X$$

is the element  $\alpha_t := \alpha(t, \cdot)$  of the given family of selfmaps of  $X$ . By moving the origin by an arbitrarily small translation, we can assume  $JF(0) \neq 0$ . We set  $\beta = \prod_{i=1}^{\infty} (1 - \varepsilon_i) > 0$  and choose  $m_0$  big enough so that for all  $m \geq m_0$  it follows that

$$(A) \quad t \in T_m, \quad JF(0) > \frac{1}{m^n \beta^n}, \quad F(0) \in B_{\frac{m+3}{2}}.$$

We fix a number  $m \geq m_0$  and claim that

$$(B) \quad \forall i \geq m + 2 \quad F(B_m) \cap \bigcup_{j=1}^{k(i)} \{a_j^i\} = \emptyset.$$

Suppose the contrary, i.e., there exists  $z \in B_m$  with  $F(z) = a_j^i$  for some  $i \geq m$  and  $1 \leq j \leq k(i)$ . Since  $F(\mathbb{C}^n \setminus \varphi(X)) \subset \mathbb{C}^n \setminus \varphi(X)$ , we have  $z \in \varphi(X)$ . Let  $x = \varphi^{-1}(z) \in \varphi^{-1}(B_m)$ , i.e.,  $F \circ \varphi(x) = a_j^i = \varphi(x_j^i)$ . This means  $\alpha_t(x) = \varphi^{-1} \circ F \circ \varphi(x) = x_j^i$ . But by (\*) we have  $x \in (\Phi_{m+1} \circ \varphi_0)^{-1}(B_{m+1})$  and, since  $t \in T_m$ , it follows that

$$\varrho(\alpha_t(x)) \leq \max_{y \in (\Phi_{m+1} \circ \varphi_0)^{-1}(B_{m+2}), t \in T_{m+2}} \varrho(\alpha(t, y)).$$

Hence, since  $i - 1 \leq m + 1$ , we have

$$\varrho(\alpha_t(x)) \leq \max_{y \in (\Phi_{i-1} \circ \varphi_0)^{-1}(B_i), t \in T_i} \varrho(\alpha(t, y)).$$

According to (3<sub>i</sub>) this, together with the choice of  $x_j^i$ , implies  $\varrho(x_j^i) > \varrho(\alpha_t(x))$  which contradicts  $x_j^i = \alpha_t(x)$ . Thus property (B) is proved.

There exists a natural number  $k$  such that  $F(B_m) \subset B_{k+2}$ . Suppose  $k > m + 2$ . We consider the holomorphic maps

$$F_j: B_1 \rightarrow B_{k+2}, \quad F_j(z) := F\left(z \cdot m \prod_{l=j+1}^k \left(1 - \frac{\varepsilon_l}{2}\right)\right), \quad j = m + 2, \dots, k.$$

For  $j = m + 2, \dots, k$  we have:

First,  $F_j(B_1) = F(B_m \prod_{l=j+1}^k (1 - \frac{\varepsilon_l}{2})) \subset F(B_m)$ . Hence, from (B) it follows that  $F_j(B_1)$  does not contain any point  $a_l^j$ ,  $l = 1, 2, \dots, k(j)$ .

Second,  $JF_j(0) = m^n \prod_{l=j+1}^k \left(1 - \frac{\varepsilon_l}{2}\right)^n JF(0) > m^n \beta^n JF(0) > 1$  is a consequence of (A).

And third, also by (A), we have  $F_j(0) = F(0) \in B_{\frac{j+1}{2}}$ .

Using (1<sub>k</sub>) for the holomorphic map  $F_k$ , these three properties imply,

$$F_k(B_{(1-\frac{\epsilon_k}{2})}) = F_{k-1}(B_1) \subset B_{k+1}.$$

Proceeding by induction from  $k$  down to  $m+2$  and using property (1<sub>j</sub>) for the holomorphic map  $F_j$  we find in the same way that  $F_j(B_1) \subset B_{j+2}$ . For  $j = m+2$  this means

$$F(B_{m \cdot \rho}) \subset F_{m+2}(B_1) \subset B_{m+4} \quad \forall m \geq m_0.$$

This growth restriction forces  $F$  to be an affine map. Since  $F$  is a non-degenerate map, it must be an affine automorphism.

From (9<sub>m</sub>) and the fact that  $\varphi_0(\tilde{X})$  osculates of order  $k$  at the points  $\varphi_0(x_i) = b_i$ ,  $i = 1, 2, \dots, n+2$  it follows that the submanifold  $\varphi(\tilde{X})$  osculates of order  $k$  at the points  $\varphi(x_i)$ ,  $i = 1, 2, \dots, n+2$ . From (8<sub>m</sub>) (and the accurate choice of  $\epsilon_m$  in the beginning of the  $m$ -th step) it follows that  $\varphi(X)$  does not osculate of order  $k$  at any other point. Since the affine automorphism  $F$  leaves  $\varphi(X)$  invariant, it is clear that  $\varphi(\tilde{X})$  (the union of the components of maximal dimension of the smooth part of  $\varphi(X)$ ) is  $F$ -invariant as well. Furthermore an affine automorphism preserves the property of osculating of order  $k$  at some point. Thus  $F$  permutes the points  $b_1, b_2, \dots, b_{n+2} \in \mathbb{C}^n$ . But no affine automorphism except the identity permutes these points. Hence  $F = \text{id}$ .  $\square$

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