Error-Correcting Codes and B_h-Sequences

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Abstract—**We construct error-correcting (nonlinear) binary codes using a construction of Bose and Chowla in additive number theory. Our method extends a construction of Graham and Sloane for constant weight codes. The new codes improve 1028 of the 7168 best known h-error-correcting codes of word length ≤ 512 with $1 \leq h \leq 14$. We give asymptotical comparisons to shortened Bose–Chaudhuri–Hocquenghem (BCH) codes.**

***Index Terms*—$A(n,d)$, $A(n,d,w)$, B_h-sequences, binary codes, tables.**

I. Introduction

Let $\mathbb{F}_2$ be the field with two elements. The following standard definitions are of particular interest to coding theory

$$A(n,d) := \max\{\#C \mid C \subseteq \mathbb{F}_2^n \text{ is a code with minimum distance } \geq d\}$$

$$A(n,d,w) := \max\{\#C \mid C \subseteq \mathbb{F}_2^n \text{ has minimum distance } \geq d \text{ and constant weight } w\}.$$

The goal of this paper is to use B_h-sequences to construct good codes. For various values of n and d we will improve the best known lower bound for $A(n,d)$.

A table of best known lower bounds for $A(n,d)$ for $n \leq 512$ and $d \leq 29$ can be found on the web pages maintained by Litsyn, Rains, and Sloane (see [7]). These web pages are based on the published tables of best known codes by MacWilliams and Sloane (see [8]) and, more recently, by Litsyn (see [6]). A table with lower bounds for $A(n,d,w)$ can be found in [8] or at the web page [9]. For many values of n and d, the constructions in this paper give better lower bounds for $A(n,d)$ than those known up to now. We supplied a table of all improved lower bounds for $A(n,d)$ in Section V.

In Section II, we discuss B_h-sequences and their relation to error-correcting codes. In Section III, we present a generalized Bose–Chowla construction for B_h-sequences. We apply the construction of B_h-sequences to obtain good lower bounds for $A(n,d)$. A similar construction was already used by Graham and Sloane for constant-weight codes (see [3]). In Section IV, we study the asymptotics of our bounds and we compare our bounds to the bounds derived from shortened Bose–Chaudhuri–Hocquenghem (BCH) codes.

II. B_h-Sequences

The notion of B_h-sequence comes from additive number theory (see, for example, [4, Ch. II]).

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Definition 1: A sequence $g_1, \ldots, g_n$ in an Abelian group G is called a B_h-sequence if all

$$g_{i_1} + g_{i_2} + \cdots + g_{i_h}, \qquad 1 \leq i_1 \leq i_2 \leq \cdots \leq i_h \leq n$$

are distinct.

We now discuss the relation between B_h-sequences and codes. For two words $x, y \in \mathbb{F}_2$ we will write $\mathrm{wt}(x)$ for the weight of x and $\delta(x,y) = \mathrm{wt}(x-y)$ for the Hamming distance between x and y.

Proposition 2: Suppose that $g_1, \ldots, g_n$ is a B_h-sequence in an Abelian group G. Identify $\mathbb{F}_2$ with $\{0,1\} \subset \mathbb{Z}$ and define the map $\phi : \mathbb{F}_2^n \to G$ by

$$\phi(x_1, \ldots, x_n) = x_1 g_1 + x_2 g_2 + \cdots + x_n g_n.$$

For every integer w and every $g \in G$ we have that

$$\phi^{-1}(g) \cap \{x \in \mathbb{F}_2^n \mid \mathrm{wt}(x) = w\}$$

is a code with minimum distance $\geq 2h+2$ for which all codewords have weight w.

Proof: Suppose that $x, y \in \mathbb{F}_2^n$ such that $\mathrm{wt}(x) = \mathrm{wt}(y) = w$, $\phi(x) = \phi(y) = g \in G$, and $\delta(x,y) < 2h+2$. We will show that $x = y$. In G we have the equality

$$x_1 g_1 + x_2 g_2 + \cdots + x_n g_n = y_1 g_1 + y_2 g_2 + \cdots + y_n g_n. \quad (1)$$

Let $X = \{i \mid x_i = 1, y_i = 0\}$ and $Y = \{i \mid x_i = 0, y_i = 1\}$. Note that $\#X = \#Y$ because $\mathrm{wt}(x) = \mathrm{wt}(y)$ and $\#X + \#Y = \delta(x,y) < 2h+2$. In particular, $\delta(x,y)$ is even and $\#X = \#Y \leq h$. Now (1) simplifies to

$$\sum_{i \in X} g_i = \sum_{i \in Y} g_i.$$

We have

$$(h - \#X) g_1 + \sum_{i \in X} g_i = (h - \#Y) g_1 + \sum_{i \in Y} g_i.$$

We conclude that $X = Y$ because $g_1, g_2, \ldots, g_n$ is a B_h-sequence. Since X and Y are obviously disjoint we get $X = Y = \emptyset$ and, therefore, $x = y$. □

Definition 3: Let $c(n,h)$ be the smallest positive integer C such that an Abelian group G with cardinality C and a B_h-sequence of cardinality n in G exist.

Corollary 4: We have the following inequality:

$$A(n, 2h+2, w) \geq \frac{1}{c(n,h)} \binom{n}{w}.$$

Proof: Suppose that $g_1, g_2, \ldots, g_n$ is a B_h-sequence in an Abelian group G with $\#G = c(n,h)$. Let $\phi : \mathbb{F}_2^n \to G$ be as in Proposition 2. By the pigeonhole principle, at least for one $g \in G$ we must have

$$\#\{x \in \mathbb{F}_2^n \mid \mathrm{wt}(x) = w\} \cap \phi^{-1}(g) \geq \frac{1}{\#G} \#\{x \in \mathbb{F}_2^n \mid \mathrm{wt}(x) = w\} = \frac{1}{\#G}\binom{n}{w}$$

and by Proposition 2 we have that

$$\{x \in \mathbb{F}_2^n \mid \mathrm{wt}(x) = w\} \cap \phi^{-1}(g)$$

is a code consisting of words with weight w having minimum distance $2h+2$. □

III. A Construction for B_h Sequences

Let q be a prime power and let $\mathbb{F}_q$ be the field with q elements. Bose and Chowla constructed B_h-sequences in the cyclic group $\mathbb{F}_q^\star$ and the integers (see [1] or [4, Sec. II, Par. 2, Theorem 3]). Suppose that $P(X)$ is a polynomial with coefficients in $\mathbb{F}_q$. We will work in the ring $\mathbb{F}_q[X]/P(X)$. The multiplicative group of invertible elements in $\mathbb{F}_q[X]/P(X)$ is denoted by $(\mathbb{F}_q[X]/P(X))^\star$. The construction of Bose and Chowla easily generalizes to the construction of B_h-sequences in the (not necessarily cyclic) group $(\mathbb{F}_q[X]/P(X))^\star$. Note that $\mathbb{F}_q^\star$, the multiplicative group of units in $\mathbb{F}_q$, is a subgroup of $(\mathbb{F}_q[X]/P(X))^\star$.

Theorem 5 : Suppose that $P(X)$ is a polynomial in $\mathbb{F}_q[X]$, and $a_1, \ldots, a_n \in \mathbb{F}_q$ are distinct with $P(a_i) \neq 0$ for all i.

a) If $P(X)$ has degree h, then the sequence

$$X - a_1, X - a_2, \ldots, X - a_n$$

is a B_h-sequence in the (multiplicative) group $(\mathbb{F}_q[X]/P(X))^\star$.

b) Suppose that $P(X)$ has degree $h+1$ then (the image of)

$$1, X - a_1, X - a_2, \ldots, X - a_n$$

is a B_h-sequence in $(\mathbb{F}_q[X]/P(X))^\star/\mathbb{F}_q^\star$.

Proof: The proof is similar to the original proof of Bose and Chowla (see [1] or [4, Sec. II, Par. 2, Theorem 3]). □

We would like to give upper bounds for $c(n,h)$ using the previous theorem. Let us define $\mu(P(X)) = \#(\mathbb{F}_q[X]/P(X))^\star$ for any polynomial $P(X) \in \mathbb{F}_q[X]$. Let us first see how we can minimalize $\mu(P(X))$ by varying the choice of $P(X)$.

Lemma 6: Let $S \subseteq \mathbb{F}_q$ be a subset with $\#S = n$ and suppose that h is a positive integer such that $h < q$. The smallest possible value of $\mu(P(X))$ where $P(X)$ is a polynomial of degree h satisfying $P(x) \neq 0$ for all $x \in S$ will be denoted by $\mu(q,n,h)$. The values of $\mu(q,n,h)$ are given in the table at the bottom of the page.

Proof: Suppose that $r \geq 1$ is a positive integer. Note that $\mathbb{F}_{q^r}$ has at least q elements that do not lie in any proper subfield. For example, one can take $\alpha, \alpha+1, \ldots, \alpha+q-1$ where $\alpha \in \mathbb{F}_{q^r}$ is a generator over the field $\mathbb{F}_q$. This implies that if $P(X)$ is a polynomial of degree $< q$ and r is a positive integer, then one can find an irreducible polynomial $Q(X) \in \mathbb{F}[X]$ of degree r such that $P(X)$ and $Q(X)$ are relatively prime. We will use this repeatedly in the proof.

If $P(X), Q(X) \in \mathbb{F}_q[X]$ are polynomials and $Q(X)$ has degree d, then

$$\mu(P(X))\mu(Q(X)) \leq \mu(P(X)Q(X)) \leq \mu(P(X))q^d.$$

The left inequality is an equality if and only if $P(X)$ and $Q(X)$ are relatively prime. The right inequality is an equality if and only if every irreducible factor of $Q(X)$ is an irreducible factor of $P(X)$.

Suppose now that $P(X)$ is a polynomial as in Lemma 6 such that $\mu(P(X))$ is minimal. First assume that $S \neq \mathbb{F}_q$.

Suppose $P(X)$ has an irreducible factor $P_1(X)$ of degree $d \geq 3$. We can write $P(X) = P_1(X)P_2(X)$. Let $b \in \mathbb{F}_q \setminus S$ and let $P_3(X)$ be an irreducible polynomial of degree $d-1$ relatively prime to $P_2(X)$.

Then we get

$$\begin{aligned}\mu(P(X)) &\geq \mu(P_1(X))\mu(P_2(X)) = (q^d - 1)\mu(P_2(X))\\ &> (q^{d-1} - 1)q\mu(P_2(X)) \geq\\ &\geq \mu(P_3(X))q\mu(P_2(X))\\ &\geq \mu(P_3(X))\mu((X-b)P_2(X))\\ &= \mu(P_3(X)(X-b)P_2(X)).\end{aligned}$$

This shows that $P(X)$ can only have linear and quadratic factors.

Suppose that $P(X)$ has a irreducible quadratic factor $P_1(X)$ of higher multiplicity. We can write $P(X) = P_1(X)^2P_2(X)$. Let $P_3(X)$ be an irreducible quadratic polynomial which is relatively prime to $P(X)$. Then we have

$$\begin{aligned}\mu(P(X)) &= q^2\mu(P_1(X)P_2(X)) > (q^2-1)\mu(P_1(X)P_2(X))\\ &= \mu(P_1(X)P_2(X)P_3(X)).\end{aligned}$$

This is a contradiction, so all quadratic factors of $P(X)$ should be distinct.

If $P(X)$ has a quadratic factor then $P(X)$ must vanish on all $x \in \mathbb{F}_q \setminus S$ because otherwise we could replace the quadratic factor by $(X-b)^2$ where $X-b$ is relative prime to $P(X)$ and get a smaller value of $\mu(P(X))$. In a similar way, it also follows that if $P(X)$ does not vanish on all $x \in \mathbb{F}_q \setminus S$, then $P(X)$ cannot have multiple zeros in $\mathbb{F}_q$ because otherwise we could replace a factor $(X-b)^2$ of $P(X)$ by $(X-b)(X-c)$ where $X-c$ is relatively prime to $P(X)$. If $P(X)$ has a zero of multiplicity ≥ 3, say $(X-b)^3$ divides $P(X)$, then we could replace the factor $(X-b)^2$ by an irreducible quadratic polynomial of degree 2 and decrease $\mu(P(X))$. This shows that $P(X)$ has zeros in $\mathbb{F}_q$ with multiplicities at most 2. Suppose that $P(X)$ has two

q,n,h	$\mu(q,n,h)$
$q \geq n+h$	$(q-1)^h$
$n < q < n+h, n+h-q$ even	$(q-1)^{q-n}(q^2-1)^{(n+h-q)/2}$
$n < q < n+h, n+h-q$ odd	$q(q-1)^{q-n}(q^2-1)^{(n+h-q-1)/2}$
$q = n, h$ even	$(q^2-1)^{h/2}$
$q = n, h$ odd	$(q^2-1)^{(h-3)/2}(q^3-1)$

distinct zeros in $\mathbb{F}_q$ with multiplicity 2, say $(X-b)^2(X-c)^2$ divides $P(X)$. We can replace the factor $(X-b)(X-c)$ by an irreducible quadratic polynomial relatively prime to $P(X)$ which would decrease $\mu(P(X))$ again. This shows that $P(X)$ has at most one zero in $\mathbb{F}_q$ with multiplicity 2.

If $q-n \geq h$ then $P(X)$ cannot have quadratic factors or multiple zeros, because $P(X)$ cannot vanish on $\mathbb{F}_q \setminus S$. In that case, $P(X)$ is a product of distinct linear factors and $\mu(P(X)) = (q-1)^h$. If $q-n<h$, then $P(X)$ must have a quadratic factor or a multiple zero in $\mathbb{F}_q$. Therefore, $P(X)$ must vanish on $\mathbb{F}_q \setminus S$. Since there is at most one zero with multiplicity 2, there are either $q-n$ or $q-n+1$ linear factors. If $h+n-q$ is even, then there are exactly $q-n$ distinct linear factors and $(h+n-q)/2$ distinct quadratic irreducible factors. In that case, we have

$$\mu(P(X)) = (q-1)^{q-n}(q^2-1)^{(h+n-q)/2}.$$

If $h+n-q$ is odd, then $P(X)$ has $q-n$ zeros of which one has multiplicity 2. The number of quadratic irreducible factors is $(h+n-q-1)/2$. In this case, we have

$$\mu(P(X)) = q(q-1)^{q-n}(q^2-1)^{(h+n-q-1)/2}.$$

Now we need to discuss the case that $S = \mathbb{F}_q$. Similar arguments as before show that now $P(X)$ cannot have factors of degree ≥ 4 and that all factors of $P(X)$ are distinct. Now $P(X)$ can have at most one irreducible factor of degree 3 because otherwise two irreducible factors of degree 3 could be replaced by three distinct irreducible factors of degree 2 which are relatively prime to $P(X)$. If h is even, then $P(X)$ has $h/2$ irreducible factors of degree 2 and then $\mu(P(X)) = (q^2-1)^{h/2}$. If h is odd, then $P(X)$ has $(h-3)/2$ irreducible factors of degree 2 and one irreducible factor of degree 3. In that case, we have

$$\mu(P(X)) = (q^2-1)^{(h-3)/2}(q^3-1). \qquad \square$$

Theorem 7: We have the following inequalities:

a) if $q = n-1$ is a prime power, then

$$c(n,h) \leq \frac{\mu(q,q,h+1)}{q-1};$$

b) if $n-1$ is not a prime power, then

$$c(n,h) \leq \mu(q,n,h)$$

where q is the smallest prime power $\geq n$.

Proof: Part a) follows from Theorem 5 b) and part b) follows from Theorem 5 a). $\square$

A lower bound for $A(n,2h+2,w)$ follows from Corollary 4, Theorem 7, and Lemma 6. This bound is almost always superior to the Gilbert–Varshamov-type bound for constant-weight codes

$$A(n,2h+2,w) \geq \frac{\binom{n}{w}}{\sum_{i=0}^{h}\binom{w}{i}\binom{n-w}{i}}. \tag{2}$$

If $C_w \subseteq \mathbb{F}_2^n$ is a code of constant weight w and minimum distance $\geq 2h+2$ for all w, and u is an integer, then

$$\bigcup_{w \equiv u (\bmod 2h+2)} C_w$$

is also a code with minimum distance $2h+2$. This shows that

$$A(n,2h+2) \geq \sum_{w \equiv u (\bmod 2h+2)} A(n,2h+2,w). \tag{3}$$

Heuristically, the best choice for u is $\lfloor n/2 \rfloor$ because

$$\sum_{w \equiv u (m \bmod 2h+2)} \binom{n}{w}$$

is maximal for $u = \lfloor n/2 \rfloor$.

Using our lower bounds for $A(n,2h+2,w)$ (or the Gilbert–Varshamov lower bound in those instances where it is better), one obtains a lower bound for $A(n,2h+2)$. For codes with odd minimum distance we note the well-known fact that

$$A(n,2h+1) = A(n+1,2h+2).$$

IV. Asymptotics

The sphere-packing bound tells us $A(n,2h+1) \leq B(n,h)$ where

$$B(n,h) = \frac{2^n}{\binom{n}{0}+\binom{n}{1}+\cdots+\binom{n}{h}} = \frac{2^n h!}{n^h}(1+o(1))$$

(as $n \to \infty$). We study the limit densities for h-error-correcting codes

$$\rho_{\inf}(h) = \liminf_{n\to\infty} \frac{A(n,2h+1)}{B(n,h)}$$

$$\text{and} \quad \rho_{\sup}(h) = \limsup_{n\to\infty} \frac{A(n,2h+1)}{B(n,h)}.$$

The best known estimates for $\rho_{\inf}(h)$ and $\rho_{\sup}(h)$ up to now came from BCH codes: a BCH code of length $n = 2^m-1$ of designed distance $2h+1$ has minimum distance $\geq 2h+1$ and dimension $\geq n-hm$, so

$$A(2^m-1,2h+1) \geq 2^{n-hm} = \frac{2^n}{(n+1)^h}$$

and it follows that

$$\rho_{\sup}(h) \geq \frac{1}{h!}.$$

If $2^{m-1} \leq n \leq 2^m-1$ then shortening the BCH code of length 2^m-1 above gives a code of length n, minimum distance $\geq 2h+1$ of dimension $\geq n-hm$. From this follows that

$$A(n,2h+1) \geq 2^{n-hm} \geq \frac{2^n}{(2n)^h}$$

and it follows that

$$\rho_{\inf}(h) \geq \frac{1}{h!2^h}.$$

TABLE I
TABLE OF IMPROVED BOUNDS FOR $A(n, d)$

n	d	new	old	ratio	n	d	new	old	ratio	n	d	new	old	ratio
279	5	261.1513	261.0000	1.1106	280	5	262.1462	262.0000	1.1067	281	5	263.1410	263.0000	1.1027
282	5	264.1257	264.0000	1.0911	283	5	265.1206	265.0000	1.0872	284	5	266.0752	266.0000	1.0535
285	5	267.0752	267.0000	1.0535	286	5	268.0752	268.0000	1.0535	287	5	269.0702	269.0000	1.0499
288	5	270.0652	270.0000	1.0462	289	5	271.0602	271.0000	1.0426	290	5	272.0354	272.0000	1.0248
291	5	273.0305	273.0000	1.0213	292	5	274.0255	274.0000	1.0179	293	5	275.0206	275.0000	1.0144
312	7	285.1299	285.0000	1.0943	313	7	286.1254	286.0000	1.0908	314	7	287.0841	287.0000	1.0600
315	7	288.0796	288.0000	1.0567	316	7	289.0750	289.0000	1.0533	317	7	290.0705	290.0000	1.0500
168	9	136.0752	136.0000	1.0535	169	9	137.0666	137.0000	1.0473	276	9	241.2231	241.0000	1.1673
277	9	242.2179	242.0000	1.1630	278	9	243.1507	243.0000	1.1101	279	9	244.1455	244.0000	1.1061
280	9	245.1404	245.0000	1.1022	281	9	246.1352	246.0000	1.0983	282	9	247.0995	247.0000	1.0714
283	9	248.0944	248.0000	1.0676	316	9	280.4447	280.0000	1.3611	317	9	281.4402	281.0000	1.3568
318	9	282.2128	282.0000	1.1589	319	9	283.2128	283.0000	1.1589	320	9	284.2128	284.0000	1.1589
321	9	285.2128	285.0000	1.1589	322	9	286.2128	286.0000	1.1589	323	9	287.2128	287.0000	1.1589
324	9	288.2128	288.0000	1.1589	325	9	289.2128	289.0000	1.1589	326	9	290.2128	290.0000	1.1589
327	9	291.2084	291.0000	1.1554	328	9	292.2041	292.0000	1.1519	329	9	293.1997	293.0000	1.1485
330	9	294.1953	294.0000	1.1450	331	9	295.1910	295.0000	1.1415	332	9	296.1088	296.0000	1.0783
333	9	297.1045	297.0000	1.0751	334	9	298.1002	298.0000	1.0720	335	9	299.0960	299.0000	1.0688
336	9	300.0917	300.0000	1.0656	337	9	301.0874	301.0000	1.0624	338	9	302.0067	302.0000	1.0046
339	9	303.0024	303.0000	1.0017	150	11	111.2378	111.1699	1.0482	151	11	112.2284	112.0000	1.1715
281	11	237.7380	237.2384	1.4138	282	11	238.6919	238.0000	1.6154	283	11	239.6868	239.0000	1.6098
284	11	240.5606	240.0000	1.4748	285	11	241.5556	241.0000	1.4698	286	11	242.5506	242.0000	1.4647
287	11	243.5456	243.0000	1.4596	288	11	244.5406	244.0000	1.4545	289	11	245.5356	245.0000	1.4495
290	11	246.4513	246.0000	1.3672	291	11	247.4463	247.0000	1.3626	292	11	248.4414	248.0000	1.3579
293	11	249.4365	249.0000	1.3533	294	11	250.1282	250.0000	1.0929	295	11	251.1282	251.0000	1.0929
296	11	252.1282	252.0000	1.0929	297	11	253.1282	253.0000	1.0929	298	11	254.1282	254.0000	1.0929
299	11	255.1282	255.0000	1.0929	300	11	256.1282	256.0000	1.0929	301	11	257.1282	257.0000	1.0929
302	11	258.1235	258.0000	1.0894	303	11	259.1188	259.0000	1.0858	304	11	260.1141	260.0000	1.0823
305	11	261.1094	261.0000	1.0788	306	11	262.1046	262.0000	1.0752	307	11	263.1000	263.0000	1.0718
308	11	264.0206	264.0000	1.0143	309	11	265.0159	265.0000	1.0111	310	11	266.0113	266.0000	1.0078
311	11	267.0066	267.0000	1.0046	312	11	267.9650	267.0000	1.9521	313	11	268.9604	268.0000	1.9459
314	11	269.8825	269.0000	1.8436	315	11	270.8780	270.0000	1.8378	316	11	271.8734	271.0000	1.8320
317	11	272.8689	272.0000	1.8262	318	11	273.5835	273.0000	1.4984	319	11	274.5835	274.0000	1.4984
320	11	275.5835	275.0000	1.4984	321	11	276.5835	276.0000	1.4984	322	11	277.5835	277.0000	1.4984
323	11	278.5835	278.0000	1.4984	324	11	279.5835	279.0000	1.4984	325	11	280.5835	280.0000	1.4984
326	11	281.5791	281.0000	1.4939	327	11	282.5747	282.0000	1.4894	328	11	283.5704	283.0000	1.4849
329	11	284.5660	284.0000	1.4804	330	11	285.5616	285.0000	1.4759	331	11	286.5573	286.0000	1.4715
332	11	287.4492	287.0000	1.3653	333	11	288.4449	288.0000	1.3612	334	11	289.4406	289.0000	1.3572
335	11	290.4364	290.0000	1.3532	336	11	291.4320	291.0000	1.3491	337	11	292.4278	292.0000	1.3452
338	11	293.3216	293.0000	1.2497	339	11	294.3174	294.0000	1.2461	340	11	295.3132	295.0000	1.2424
341	11	296.3090	296.0000	1.2388	342	11	297.3047	297.0000	1.2352	343	11	298.3006	298.0000	1.2316
344	11	299.2294	299.0000	1.1724	345	11	300.2253	300.0000	1.1690	346	11	301.2211	301.0000	1.1656
347	11	302.2170	302.0000	1.1623	348	11	303.1796	303.0000	1.1326	349	11	304.1755	304.0000	1.1294
350	11	305.1056	305.0000	1.0760	351	11	306.1015	306.0000	1.0729	352	11	307.0974	307.0000	1.0699
353	11	308.0934	308.0000	1.0669	157	13	110.4671	110.1699	1.2288	158	13	111.1846	111.0000	1.1365
159	13	112.1758	112.0000	1.1296	160	13	113.1646	113.0000	1.1208	161	13	114.1557	114.0000	1.1140
162	13	115.1446	115.0000	1.1054	163	13	116.1357	116.0000	1.0986	281	13	229.3832	229.2384	1.1056
282	13	230.3269	230.0000	1.2543	283	13	231.3218	231.0000	1.2498	284	13	232.1651	232.0000	1.1213
285	13	233.1601	233.0000	1.1174	286	13	234.1550	234.0000	1.1135	287	13	235.1500	235.0000	1.1096
288	13	236.1450	236.0000	1.1057	289	13	237.1400	236.2384	1.8681	290	13	238.0357	237.0000	2.0502
291	13	239.0308	238.0000	2.0432	292	13	240.0258	239.0000	2.0361	293	13	241.0209	240.0000	2.0291
294	13	241.6499	241.0000	1.5690	295	13	242.6499	242.0000	1.5690	296	13	243.6498	243.0000	1.5690
297	13	244.6498	244.0000	1.5690	298	13	245.6497	245.0000	1.5689	299	13	246.6497	246.0000	1.5689
300	13	247.6497	247.0000	1.5688	301	13	248.6450	248.0000	1.5637	302	13	249.6402	249.0000	1.5585
303	13	250.6355	250.0000	1.5535	304	13	251.6307	251.0000	1.5484	305	13	252.6260	252.0000	1.5433
306	13	253.6213	253.0000	1.5382	307	13	254.6166	254.0000	1.5332	308	13	255.5184	255.0000	1.4324
309	13	256.5138	256.0000	1.4278	310	13	257.5091	257.0000	1.4232	311	13	258.5044	258.0000	1.4186
312	13	259.4536	259.0000	1.3694	313	13	260.4489	260.0000	1.3650	314	13	261.3527	261.0000	1.2769
315	13	262.3481	262.0000	1.2729	316	13	263.3436	263.0000	1.2689	317	13	264.3390	264.0000	1.2649
328	13	274.9779	274.0000	1.9697	329	13	275.9736	275.0000	1.9637	330	13	276.9692	276.0000	1.9577
331	13	277.9648	277.0000	1.9518	332	13	278.8308	278.0000	1.7786	333	13	279.8265	279.0000	1.7734

TABLE I (*Continued*)
TABLE OF IMPROVED BOUNDS FOR $A(n, d)$

n	d	new	old	ratio	n	d	new	old	ratio	n	d	new	old	ratio
334	13	280.8222	280.0000	1.7681	335	13	281.8179	281.0000	1.7628	336	13	282.8136	282.0000	1.7576
337	13	283.8093	283.0000	1.7524	338	13	284.6776	284.0000	1.5995	339	13	285.6734	285.0000	1.5948
340	13	286.6692	286.0000	1.5902	341	13	287.6650	287.0000	1.5855	342	13	288.6608	288.0000	1.5809
343	13	289.6565	289.0000	1.5763	344	13	290.5687	290.0000	1.4832	345	13	291.5645	291.0000	1.4789
346	13	292.5603	292.0000	1.4746	347	13	293.5562	293.0000	1.4704	348	13	294.5106	294.0000	1.4246
349	13	295.5064	295.0000	1.4205	350	13	296.4201	296.0000	1.3380	351	13	297.4160	297.0000	1.3342
352	13	298.4119	298.0000	1.3304	353	13	299.4078	299.0000	1.3267	354	13	300.2821	300.0000	1.2159
355	13	301.2780	301.0000	1.2125	356	13	302.2740	302.0000	1.2092	357	13	303.2700	303.0000	1.2058
358	13	304.2659	304.0000	1.2024	359	13	305.2619	305.0000	1.1991	360	13	306.2178	306.0000	1.1630
361	13	307.2138	307.0000	1.1598	362	13	308.0909	308.0000	1.0650	363	13	309.0869	309.0000	1.0621
364	13	310.0830	310.0000	1.0592	365	13	311.0791	311.0000	1.0563	366	13	312.0751	312.0000	1.0534
367	13	313.0712	313.0000	1.0506	164	15	109.4423	109.1699	1.2078	165	15	110.4337	110.0000	1.3507
166	15	111.4209	111.0000	1.3388	167	15	112.4124	112.0000	1.3309	168	15	113.2968	113.0000	1.2284
169	15	114.2883	114.0000	1.2212	170	15	115.0735	115.0000	1.0523	171	15	116.0652	116.0000	1.0462
172	15	117.0531	117.0000	1.0375	173	15	118.0449	118.0000	1.0316	289	15	228.7809	228.2384	1.4565
290	15	229.6566	229.0000	1.5763	291	15	230.6516	230.0000	1.5710	292	15	231.6463	231.0000	1.5652
293	15	232.6414	232.0000	1.5599	294	15	233.2075	233.0000	1.1547	295	15	234.2075	234.0000	1.1547
296	15	235.2072	235.0000	1.1544	297	15	236.2072	235.2384	1.9571	298	15	237.2068	236.0000	2.3083
299	15	238.2068	237.0000	2.3083	300	15	239.2018	238.0000	2.3002	301	15	240.1971	239.0000	2.2928
302	15	241.1921	240.0000	2.2848	303	15	242.1874	241.0000	2.2774	304	15	243.1824	242.0000	2.2695
305	15	244.1777	243.0000	2.2621	306	15	245.1727	244.0000	2.2543	307	15	246.1680	245.0000	2.2470
308	15	247.0510	246.0000	2.0719	309	15	248.0463	247.0000	2.0653	310	15	249.0414	248.0000	2.0582
311	15	250.0368	249.0000	2.0516	312	15	250.9764	250.0000	1.9676	313	15	251.9718	251.0000	1.9613
314	15	252.8570	252.0000	1.8113	315	15	253.8525	253.0000	1.8056	316	15	254.8477	254.0000	1.7996
317	15	255.8431	255.0000	1.7940	318	15	256.4415	256.0000	1.3581	319	15	257.4415	257.0000	1.3581
320	15	258.4413	258.0000	1.3578	321	15	259.4413	259.0000	1.3578	322	15	260.4411	260.0000	1.3576
323	15	261.4411	261.0000	1.3576	324	15	262.4365	262.0000	1.3534	325	15	263.4322	263.0000	1.3493
326	15	264.4276	264.0000	1.3450	327	15	265.4233	265.0000	1.3410	328	15	266.4187	265.0000	2.6735
329	15	267.4144	266.0000	2.6654	330	15	268.4098	267.0000	2.6570	331	15	269.4055	268.0000	2.6490
332	15	270.2453	269.0000	2.3708	333	15	271.2411	270.0000	2.3637	334	15	272.2366	271.0000	2.3565
335	15	273.2323	272.0000	2.3495	336	15	274.2279	273.0000	2.3422	337	15	275.2236	274.0000	2.3353
338	15	276.0663	275.0000	2.0941	339	15	277.0621	276.0000	2.0880	340	15	278.0578	277.0000	2.0817
341	15	279.0536	278.0000	2.0757	342	15	280.0492	279.0000	2.0694	343	15	281.0450	280.0000	2.0634
344	15	281.9403	281.0000	1.9189	345	15	282.9362	282.0000	1.9134	346	15	283.9318	283.0000	1.9077
347	15	284.9277	284.0000	1.9023	348	15	285.8737	285.0000	1.8323	349	15	286.8696	286.0000	1.8271
350	15	287.7667	287.0000	1.7013	351	15	288.7626	288.0000	1.6965	352	15	289.7584	289.0000	1.6916
353	15	290.7543	290.0000	1.6868	354	15	291.6041	291.0000	1.5200	355	15	292.6001	292.0000	1.5158
356	15	293.5960	293.0000	1.5115	357	15	294.5920	294.0000	1.5073	358	15	295.5878	295.0000	1.5030
359	15	296.5838	296.0000	1.4988	360	15	297.5316	297.0000	1.4455	361	15	298.5276	298.0000	1.4416
362	15	299.3808	299.0000	1.3020	363	15	300.3768	300.0000	1.2985	364	15	301.3728	301.0000	1.2949
365	15	302.3689	302.0000	1.2914	366	15	303.3649	303.0000	1.2878	367	15	304.3609	304.0000	1.2843
368	15	305.2165	305.0000	1.1619	369	15	306.2126	306.0000	1.1588	370	15	307.2087	307.0000	1.1556
371	15	308.2048	308.0000	1.1525	372	15	309.2008	309.0000	1.1494	373	15	310.1970	310.0000	1.1463
374	15	311.0548	311.0000	1.0387	375	15	312.0510	312.0000	1.0360	376	15	313.0471	313.0000	1.0332
377	15	314.0433	314.0000	1.0305	378	15	315.0395	315.0000	1.0277	379	15	316.0357	316.0000	1.0250
171	17	108.5553	108.1699	1.3062	172	17	109.5415	109.0000	1.4555	173	17	110.5331	110.0000	1.4471
174	17	111.1749	111.0000	1.1289	175	17	112.1668	112.0000	1.1226	176	17	113.1536	113.0000	1.1123
177	17	114.1455	114.0000	1.1061	178	17	115.1324	115.0000	1.0961	179	17	116.1243	116.0000	1.0900
192	17	128.2315	128.0000	1.1741	193	17	129.2240	129.0000	1.1680	194	17	130.0053	130.0000	1.0037
289	17	220.4594	220.2384	1.1655	290	17	221.3146	221.0000	1.2437	291	17	222.3097	222.0000	1.2394
292	17	223.3038	223.0000	1.2344	293	17	224.2988	224.0000	1.2301	297	17	227.8008	227.2384	1.4767
298	17	228.7999	228.0000	1.7410	299	17	229.7952	229.0000	1.7353	300	17	230.7896	230.0000	1.7286
301	17	231.7849	231.0000	1.7230	302	17	232.7794	232.0000	1.7164	303	17	233.7747	233.0000	1.7108
304	17	234.7692	234.0000	1.7043	305	17	235.7645	234.2384	2.8800	306	17	236.7590	235.0000	3.3846
307	17	237.7543	236.0000	3.3736	308	17	238.6181	237.0000	3.0697	309	17	239.6135	238.0000	3.0598
310	17	240.6081	239.0000	3.0485	311	17	241.6034	240.0000	3.0386	312	17	242.5334	241.0000	2.8946
313	17	243.5287	242.0000	2.8853	314	17	244.3952	243.0000	2.6303	315	17	245.3907	244.0000	2.6220
316	17	246.3854	245.0000	2.6125	317	17	247.3809	246.0000	2.6042	318	17	247.9209	247.0000	1.8932
319	17	248.9209	248.0000	1.8932	320	17	249.9202	249.0000	1.8924	321	17	250.9202	250.0000	1.8924
322	17	251.9196	251.0000	1.8916	323	17	252.9152	252.0000	1.8859	324	17	253.9103	253.0000	1.8794

TABLE I (*Continued*)
TABLE OF IMPROVED BOUNDS FOR $A(n, d)$

n	d	new	old	ratio	n	d	new	old	ratio	n	d	new	old	ratio
325	17	254.9059	254.0000	1.8737	326	17	255.9010	255.0000	1.8673	327	17	256.8966	256.0000	1.8617
328	17	257.8917	257.0000	1.8553	329	17	258.8873	258.0000	1.8497	330	17	259.8824	259.0000	1.8435
331	17	260.8780	260.0000	1.8379	332	17	261.6917	260.0000	3.2303	333	17	262.6874	261.0000	3.2207
334	17	263.6826	262.0000	3.2100	335	17	264.6783	263.0000	3.2005	336	17	265.6735	264.0000	3.1899
337	17	266.6692	265.0000	3.1804	338	17	267.4862	266.0000	2.8015	339	17	268.4820	267.0000	2.7933
340	17	269.4773	268.0000	2.7843	341	17	270.4731	269.0000	2.7761	342	17	271.4684	270.0000	2.7672
343	17	272.4642	271.0000	2.7591	344	17	273.3425	272.0000	2.5359	345	17	274.3383	273.0000	2.5286
346	17	275.3337	274.0000	2.5205	347	17	276.3296	275.0000	2.5133	348	17	277.2670	276.0000	2.4066
349	17	278.2628	277.0000	2.3997	350	17	279.1432	278.0000	2.2087	351	17	280.1391	279.0000	2.2025
352	17	281.1347	280.0000	2.1957	353	17	282.1305	281.0000	2.1894	354	17	282.9558	282.0000	1.9397
355	17	283.9518	283.0000	1.9343	356	17	284.9474	284.0000	1.9284	357	17	285.9434	285.0000	1.9230
358	17	286.9390	286.0000	1.9172	359	17	287.9350	287.0000	1.9119	360	17	288.8745	288.0000	1.8334
361	17	289.8705	289.0000	1.8283	362	17	290.6997	290.0000	1.6241	363	17	291.6957	291.0000	1.6197
364	17	292.6915	292.0000	1.6149	365	17	293.6876	293.0000	1.6105	366	17	294.6833	294.0000	1.6058
367	17	295.6794	295.0000	1.6014	368	17	296.5113	296.0000	1.4253	369	17	297.5074	297.0000	1.4215
370	17	298.5033	298.0000	1.4174	371	17	299.4994	299.0000	1.4136	372	17	300.4952	300.0000	1.4096
373	17	301.4914	301.0000	1.4058	374	17	302.3260	302.0000	1.2535	375	17	303.3222	303.0000	1.2502
376	17	304.3181	304.0000	1.2467	377	17	305.3143	305.0000	1.2434	378	17	306.3102	306.0000	1.2399
379	17	307.3064	307.0000	1.2366	380	17	308.1963	308.0000	1.1458	381	17	309.1926	309.0000	1.1428
382	17	310.1886	310.0000	1.1396	383	17	311.1848	311.0000	1.1366	384	17	312.0237	312.0000	1.0166
385	17	313.0200	313.0000	1.0140	386	17	314.0161	314.0000	1.0112	387	17	315.0124	315.0000	1.0086
388	17	316.0085	316.0000	1.0059	389	17	317.0047	317.0000	1.0033	290	19	213.0107	213.0000	1.0074
291	19	214.0057	214.0000	1.0040	297	19	219.4318	219.2384	1.1434	298	19	220.4254	220.0000	1.3429
299	19	221.4207	221.0000	1.3386	300	19	222.4144	222.0000	1.3327	301	19	223.4097	223.0000	1.3284
302	19	224.4034	224.0000	1.3226	303	19	225.3987	225.0000	1.3183	304	19	226.3924	226.0000	1.3126
305	19	227.3877	226.2384	2.2181	306	19	228.3815	227.0000	2.6053	307	19	229.3768	228.0000	2.5969
308	19	230.2212	229.0000	2.3314	309	19	231.2165	230.0000	2.3239	310	19	232.2104	231.0000	2.3141
311	19	233.2058	232.0000	2.3067	312	19	234.1258	233.0000	2.1822	313	19	235.1212	234.0000	2.1753
314	19	235.9686	235.0000	1.9570	315	19	236.9641	236.0000	1.9508	316	19	237.9581	237.0000	1.9428
317	19	238.9536	238.0000	1.9367	318	19	239.4349	239.0000	1.3518	319	19	240.4349	240.0000	1.3518
320	19	241.4336	240.0000	2.7013	321	19	242.4336	241.0000	2.7013	322	19	243.4280	242.0000	2.6908
323	19	244.4237	243.0000	2.6827	324	19	245.4181	244.0000	2.6723	325	19	246.4137	245.0000	2.6642
326	19	247.4081	246.0000	2.6540	327	19	248.4038	247.0000	2.6460	328	19	249.3983	248.0000	2.6358
329	19	250.3939	249.0000	2.6279	330	19	251.3884	250.0000	2.6178	331	19	252.3840	251.0000	2.6100
332	19	253.1711	252.0000	2.2519	333	19	254.1669	253.0000	2.2452	334	19	255.1615	254.0000	2.2369
335	19	256.1572	255.0000	2.2303	336	19	257.1518	256.0000	2.2220	337	19	258.1476	257.0000	2.2154
338	19	258.9385	258.0000	1.9165	339	19	259.9343	259.0000	1.9110	340	19	260.9291	260.0000	1.9040
341	19	261.9249	261.0000	1.8985	342	19	262.9197	262.0000	1.8917	343	19	263.9155	263.0000	1.8862
344	19	264.7765	264.0000	1.7129	345	19	265.7723	265.0000	1.7080	346	19	266.7672	266.0000	1.7020
347	19	267.7631	267.0000	1.6971	348	19	268.6916	268.0000	1.6151	349	19	269.6875	269.0000	1.6105
350	19	270.5510	270.0000	1.4651	351	19	271.5469	271.0000	1.4609	352	19	272.5419	272.0000	1.4559
353	19	273.5378	273.0000	1.4518	354	19	274.3383	274.0000	1.2642	355	19	275.3343	275.0000	1.2607
356	19	276.3294	276.0000	1.2565	357	19	277.3254	277.0000	1.2530	358	19	278.3205	278.0000	1.2488
359	19	279.3165	279.0000	1.2453	360	19	280.2476	280.0000	1.1872	361	19	281.2436	281.0000	1.1840
362	19	282.0485	282.0000	1.0342	363	19	283.0446	283.0000	1.0314	364	19	284.0399	284.0000	1.0280
365	19	285.0360	285.0000	1.0253	366	19	286.0313	286.0000	1.0219	367	19	287.0274	287.0000	1.0192
384	19	303.2843	303.0000	1.2178	385	19	304.2806	304.0000	1.2147	386	19	305.2763	305.0000	1.2111
387	19	306.2726	306.0000	1.2080	388	19	307.2683	307.0000	1.2044	389	19	308.2646	308.0000	1.2013
390	19	309.0252	309.0000	1.0176	391	19	310.0216	310.0000	1.0151	392	19	311.0174	311.0000	1.0122
393	19	312.0138	312.0000	1.0096	394	19	313.0096	313.0000	1.0067	395	19	314.0060	314.0000	1.0042
396	19	315.0018	315.0000	1.0013	298	21	212.0855	212.0000	1.0610	299	21	213.0808	213.0000	1.0576
300	21	214.0736	214.0000	1.0523	301	21	215.0689	215.0000	1.0489	302	21	216.0618	216.0000	1.0437
303	21	217.0571	217.0000	1.0403	304	21	218.0500	218.0000	1.0352	305	21	219.0453	218.2384	1.7494
306	21	220.0382	219.0000	2.0537	307	21	221.0335	220.0000	2.0470	308	21	221.8584	221.0000	1.8131
309	21	222.8538	222.0000	1.8072	310	21	223.8469	223.0000	1.7986	311	21	224.8422	224.0000	1.7928
312	21	225.7522	225.0000	1.6843	313	21	226.7475	225.2384	2.8464	314	21	227.5759	226.0000	2.9812
315	21	228.5713	227.0000	2.9718	316	21	229.5647	228.0000	2.9581	317	21	230.5601	229.0000	2.9487
318	21	230.9827	230.0000	1.9761	319	21	231.9827	231.0000	1.9761	320	21	232.9806	232.0000	1.9733
321	21	233.9762	232.2384	3.3353	322	21	234.9699	233.0000	3.9173	323	21	235.9655	234.0000	3.9055
324	21	236.9592	235.0000	3.8884	325	21	237.9548	236.0000	3.8766	326	21	238.9485	237.0000	3.8597

TABLE I (*Continued*)
TABLE OF IMPROVED BOUNDS FOR $A(n, d)$

n	d	new	old	ratio	n	d	new	old	ratio	n	d	new	old	ratio
327	21	239.9441	238.0000	3.8481	328	21	240.9379	239.0000	3.8314	329	21	241.9335	240.0000	3.8198
330	21	242.9273	241.0000	3.8034	331	21	243.9229	242.0000	3.7919	332	21	244.6834	243.0000	3.2118
333	21	245.6791	244.0000	3.2023	334	21	246.6730	245.0000	3.1889	335	21	247.6688	246.0000	3.1794
336	21	248.6627	247.0000	3.1661	337	21	249.6584	248.0000	3.1567	338	21	250.4232	249.0000	2.6818
339	21	251.4190	250.0000	2.6740	340	21	252.4131	251.0000	2.6631	341	21	253.4089	252.0000	2.6553
342	21	254.4030	253.0000	2.6445	343	21	255.3988	254.0000	2.6368	344	21	256.2424	255.0000	2.3659
345	21	257.2383	256.0000	2.3591	346	21	258.2325	257.0000	2.3497	347	21	259.2283	258.0000	2.3429
348	21	260.1480	259.0000	2.2161	349	21	261.1439	260.0000	2.2097	350	21	261.9902	261.0000	1.9865
351	21	262.9861	262.0000	1.9809	352	21	263.9805	263.0000	1.9732	353	21	264.9764	264.0000	1.9676
354	21	265.7520	265.0000	1.6841	355	21	266.7479	266.0000	1.6794	356	21	267.7425	267.0000	1.6730
357	21	268.7385	268.0000	1.6684	358	21	269.7330	269.0000	1.6621	359	21	270.7290	270.0000	1.6575
360	21	271.6515	271.0000	1.5708	361	21	272.6475	272.0000	1.5664	362	21	273.4280	273.0000	1.3454
363	21	274.4241	274.0000	1.3417	364	21	275.4188	275.0000	1.3368	365	21	276.4149	276.0000	1.3332
366	21	277.4096	277.0000	1.3283	367	21	278.4057	278.0000	1.3247	368	21	279.1898	279.0000	1.1406
369	21	280.1860	280.0000	1.1376	370	21	281.1808	281.0000	1.1335	371	21	282.1770	282.0000	1.1305
372	21	283.1718	283.0000	1.1265	373	21	284.1680	284.0000	1.1235	384	21	294.5738	294.0000	1.4884
385	21	295.5700	295.0000	1.4846	386	21	296.5653	296.0000	1.4796	387	21	297.5615	297.0000	1.4758
388	21	298.5568	298.0000	1.4710	389	21	299.5530	299.0000	1.4672	390	21	300.2838	300.0000	1.2174
391	21	301.2802	301.0000	1.2144	392	21	302.2755	302.0000	1.2105	393	21	303.2719	303.0000	1.2074
394	21	304.2673	304.0000	1.2035	395	21	305.2636	305.0000	1.2005	396	21	306.2590	306.0000	1.1967
397	21	307.2554	307.0000	1.1937	398	21	308.1206	308.0000	1.0872	399	21	309.1170	309.0000	1.0845
400	21	310.1125	310.0000	1.0811	401	21	311.1089	311.0000	1.0784	305	23	210.7313	210.2384	1.4072
306	23	211.7235	211.0000	1.6511	307	23	212.7188	212.0000	1.6458	308	23	213.5242	213.0000	1.4382
309	23	214.5196	214.0000	1.4336	310	23	215.5119	215.0000	1.4260	311	23	216.5073	216.0000	1.4214
312	23	217.4072	217.0000	1.3262	313	23	218.4027	217.2384	2.2412	314	23	219.2119	218.0000	2.3164
315	23	220.2074	219.0000	2.3092	316	23	221.1999	220.0000	2.2973	317	23	222.1954	221.0000	2.2901
318	23	222.5592	222.0000	1.4735	319	23	223.5592	223.0000	1.4735	320	23	224.5521	224.0000	1.4662
321	23	225.5477	224.2384	2.4782	322	23	226.5406	225.0000	2.9091	323	23	227.5362	226.0000	2.9004
324	23	228.5292	227.0000	2.8862	325	23	229.5248	228.0000	2.8775	326	23	230.5178	229.0000	2.8635
327	23	231.5134	230.0000	2.8549	328	23	232.5064	231.0000	2.8411	329	23	233.5021	231.2384	4.8021
330	23	234.4951	232.0000	5.6377	331	23	235.4908	233.0000	5.6208	332	23	236.2246	234.0000	4.6738
333	23	237.2203	235.0000	4.6600	334	23	238.2135	236.0000	4.6381	335	23	239.2093	237.0000	4.6244
336	23	240.2025	238.0000	4.6027	337	23	241.1982	239.0000	4.5892	338	23	241.9368	240.0000	3.8287
339	23	242.9326	241.0000	3.8175	340	23	243.9260	242.0000	3.8001	341	23	244.9218	243.0000	3.7890
342	23	245.9153	244.0000	3.7718	343	23	246.9111	245.0000	3.7609	344	23	247.7373	246.0000	3.3340
345	23	248.7331	247.0000	3.3244	346	23	249.7266	248.0000	3.3096	347	23	250.7225	249.0000	3.3001
348	23	251.6332	250.0000	3.1020	349	23	252.6291	251.0000	3.0932	350	23	253.4583	252.0000	2.7479
351	23	254.4542	253.0000	2.7401	352	23	255.4479	254.0000	2.7282	353	23	256.4439	255.0000	2.7205
354	23	257.1944	256.0000	2.2885	355	23	258.1904	257.0000	2.2822	356	23	259.1843	258.0000	2.2725
357	23	260.1803	259.0000	2.2662	358	23	261.1741	260.0000	2.2566	359	23	262.1701	261.0000	2.2503
360	23	263.0839	262.0000	2.1198	361	23	264.0799	263.0000	2.1140	362	23	264.8360	264.0000	1.7852
363	23	265.8321	265.0000	1.7803	364	23	266.8262	266.0000	1.7730	365	23	267.8223	267.0000	1.7682
366	23	268.8164	268.0000	1.7610	367	23	269.8125	269.0000	1.7562	368	23	270.5726	270.0000	1.4872
369	23	271.5687	271.0000	1.4832	370	23	272.5630	272.0000	1.4773	371	23	273.5591	273.0000	1.4733
372	23	274.5534	274.0000	1.4675	373	23	275.5495	275.0000	1.4636	374	23	276.3135	276.0000	1.2427
375	23	277.3097	277.0000	1.2395	376	23	278.3041	278.0000	1.2346	377	23	279.3003	279.0000	1.2314
378	23	280.2947	280.0000	1.2266	379	23	281.2909	281.0000	1.2234	380	23	282.1339	282.0000	1.0972
381	23	283.1301	283.0000	1.0944	382	23	284.1246	284.0000	1.0902	383	23	285.1209	285.0000	1.0874
384	23	285.8911	285.0000	1.8546	385	23	286.8874	286.0000	1.8498	386	23	287.8820	287.0000	1.8429
387	23	288.8783	288.0000	1.8382	388	23	289.8729	289.0000	1.8314	389	23	290.8693	290.0000	1.8267
390	23	291.5701	291.0000	1.4846	391	23	292.5665	292.0000	1.4809	392	23	293.5612	293.0000	1.4755
393	23	294.5576	294.0000	1.4718	394	23	295.5524	295.0000	1.4665	395	23	296.5488	296.0000	1.4628
396	23	297.5436	297.0000	1.4576	397	23	298.5400	298.0000	1.4539	398	23	299.3902	299.0000	1.3106
399	23	300.3866	300.0000	1.3073	400	23	301.3815	301.0000	1.3027	401	23	302.3779	302.0000	1.2995
402	23	303.0877	303.0000	1.0627	403	23	304.0842	304.0000	1.0601	404	23	305.0792	305.0000	1.0565
405	23	306.0757	306.0000	1.0539	406	23	307.0708	307.0000	1.0503	407	23	308.0672	308.0000	1.0477
408	23	309.0623	309.0000	1.0441	409	23	310.0588	310.0000	1.0416	305	25	202.4382	202.2384	1.1485
306	25	203.4298	203.0000	1.3470	307	25	204.4250	204.0000	1.3426	308	25	205.2113	205.0000	1.1577
309	25	206.2066	206.0000	1.1540	310	25	207.1984	207.0000	1.1474	311	25	208.1937	208.0000	1.1437
312	25	209.0838	209.0000	1.0598	313	25	210.0792	209.2384	1.7910	314	25	210.8696	210.0000	1.8271

TABLE I (*Continued*)
TABLE OF IMPROVED BOUNDS FOR $A(n, d)$

n	d	new	old	ratio	n	d	new	old	ratio	n	d	new	old	ratio
315	25	211.8650	211.0000	1.8214	316	25	212.8570	212.0000	1.8113	317	25	213.8524	213.0000	1.8055
318	25	214.1577	214.0000	1.1155	319	25	215.1533	215.0000	1.1121	320	25	216.1456	216.0000	1.1062
321	25	217.1412	216.2384	1.8697	322	25	218.1335	217.0000	2.1939	323	25	219.1291	218.0000	2.1872
324	25	220.1215	219.0000	2.1757	325	25	221.1171	220.0000	2.1691	326	25	222.1095	221.0000	2.1577
327	25	223.1051	222.0000	2.1511	328	25	224.0975	223.0000	2.1399	329	25	225.0932	223.2384	3.6169
330	25	226.0856	224.0000	4.2445	331	25	227.0812	225.0000	4.2317	332	25	227.7886	226.0000	3.4547
333	25	228.7843	227.0000	3.4445	334	25	229.7769	228.0000	3.4269	335	25	230.7726	229.0000	3.4167
336	25	231.7653	230.0000	3.3994	337	25	232.7610	230.2384	5.7460	338	25	233.4735	231.0000	5.5540
339	25	234.4693	232.0000	5.5378	340	25	235.4621	233.0000	5.5103	341	25	236.4579	234.0000	5.4942
342	25	237.4508	235.0000	5.4670	343	25	238.4465	236.0000	5.4510	344	25	239.2554	237.0000	4.7747
345	25	240.2513	238.0000	4.7610	346	25	241.2442	239.0000	4.7378	347	25	242.2400	240.0000	4.7241
348	25	243.1419	241.0000	4.4134	349	25	244.1377	242.0000	4.4007	350	25	244.9499	243.0000	3.8636
351	25	245.9459	244.0000	3.8527	352	25	246.9390	245.0000	3.8344	353	25	247.9349	246.0000	3.8235
354	25	248.6605	247.0000	3.1614	355	25	249.6565	248.0000	3.1525	356	25	250.6498	249.0000	3.1379
357	25	251.6458	250.0000	3.1292	358	25	252.6391	251.0000	3.1147	359	25	253.6351	252.0000	3.1060
360	25	254.5403	253.0000	2.9085	361	25	255.5363	254.0000	2.9004	362	25	256.2680	255.0000	2.4083
363	25	257.2641	256.0000	2.4017	364	25	258.2576	257.0000	2.3910	365	25	259.2537	258.0000	2.3844
366	25	260.2472	259.0000	2.3738	367	25	261.2432	260.0000	2.3673	368	25	261.9794	261.0000	1.9717
369	25	262.9755	262.0000	1.9664	370	25	263.9692	263.0000	1.9578	371	25	264.9653	264.0000	1.9525
372	25	265.9590	265.0000	1.9440	373	25	266.9551	266.0000	1.9388	374	25	267.6956	267.0000	1.6195
375	25	268.6918	268.0000	1.6153	376	25	269.6856	269.0000	1.6084	377	25	270.6818	270.0000	1.6041
378	25	271.6756	271.0000	1.5973	379	25	272.6718	272.0000	1.5931	380	25	273.4991	273.0000	1.4133
381	25	274.4953	274.0000	1.4096	382	25	275.4893	275.0000	1.4037	383	25	276.4855	276.0000	1.4001
384	25	277.2327	276.0000	2.3501	385	25	278.2290	277.0000	2.3441	386	25	279.2231	278.0000	2.3345
387	25	280.2194	279.0000	2.3285	388	25	281.2135	280.0000	2.3189	389	25	282.2097	281.0000	2.3130
390	25	282.8807	282.0000	1.8412	391	25	283.8770	283.0000	1.8366	392	25	284.8713	284.0000	1.8293
393	25	285.8676	285.0000	1.8247	394	25	286.8619	286.0000	1.8174	395	25	287.8583	287.0000	1.8128
396	25	288.8525	288.0000	1.8057	397	25	289.8489	289.0000	1.8011	398	25	290.6841	290.0000	1.6067
399	25	291.6805	291.0000	1.6027	400	25	292.6749	292.0000	1.5965	401	25	293.6713	293.0000	1.5925
402	25	294.3521	294.0000	1.2764	403	25	295.3485	295.0000	1.2733	404	25	296.3430	296.0000	1.2684
405	25	297.3395	297.0000	1.2653	406	25	298.3340	298.0000	1.2605	407	25	299.3305	299.0000	1.2574
408	25	300.3250	300.0000	1.2527	409	25	301.3215	301.0000	1.2496	306	27	195.1500	195.0000	1.1095
307	27	196.1453	196.0000	1.1060	313	27	201.7703	201.2384	1.4458	314	27	202.5418	202.0000	1.4558
315	27	203.5373	203.0000	1.4513	316	27	204.5288	204.0000	1.4427	317	27	205.5243	205.0000	1.4382
321	27	208.7498	208.2384	1.4255	322	27	209.7417	209.0000	1.6721	323	27	210.7373	210.0000	1.6671
324	27	211.7292	211.0000	1.6577	325	27	212.7248	212.0000	1.6527	326	27	213.7168	213.0000	1.6435
327	27	214.7124	214.0000	1.6385	328	27	215.7044	215.0000	1.6294	329	27	216.7000	215.2384	2.7541
330	27	217.6920	216.0000	3.2310	331	27	218.6877	217.0000	3.2213	332	27	219.3686	218.0000	2.5822
333	27	220.3643	219.0000	2.5746	334	27	221.3565	220.0000	2.5606	335	27	222.3522	221.0000	2.5530
336	27	223.3444	222.0000	2.5392	337	27	224.3401	222.2384	4.2921	338	27	225.0267	223.0000	4.0748
339	27	226.0225	224.0000	4.0630	340	27	227.0149	225.0000	4.0414	341	27	228.0107	226.0000	4.0297
342	27	229.0030	227.0000	4.0084	343	27	229.9988	228.0000	3.9968	344	27	230.7905	229.0000	3.4594
345	27	231.7864	229.2384	5.8480	346	27	232.7788	230.0000	6.8630	347	27	233.7747	231.0000	6.8433
348	27	234.6677	232.0000	6.3543	349	27	235.6636	233.0000	6.3362	350	27	236.4589	234.0000	5.4980
351	27	237.4548	235.0000	5.4824	352	27	238.4475	236.0000	5.4546	353	27	239.4434	237.0000	5.4392
354	27	240.1442	238.0000	4.4206	355	27	241.1402	239.0000	4.4083	356	27	242.1330	240.0000	4.3863
357	27	243.1290	241.0000	4.3742	358	27	244.1218	242.0000	4.3525	359	27	245.1178	243.0000	4.3404
360	27	246.0145	244.0000	4.0405	361	27	247.0105	245.0000	4.0293	362	27	247.7180	246.0000	3.2898
363	27	248.7141	247.0000	3.2809	364	27	249.7071	248.0000	3.2651	365	27	250.7032	249.0000	3.2562
366	27	251.6962	250.0000	3.2405	367	27	252.6923	251.0000	3.2317	368	27	253.4046	252.0000	2.6474
369	27	254.4007	253.0000	2.6403	370	27	255.3939	254.0000	2.6279	371	27	256.3900	255.0000	2.6209
372	27	257.3832	256.0000	2.6085	373	27	258.3794	257.0000	2.6016	374	27	259.0963	258.0000	2.1381
375	27	260.0925	259.0000	2.1324	376	27	261.0858	260.0000	2.1226	377	27	262.0820	261.0000	2.1170
378	27	263.0754	262.0000	2.1073	379	27	264.0716	263.0000	2.1017	380	27	264.8832	264.0000	1.8445
381	27	265.8795	265.0000	1.8397	382	27	266.8729	266.0000	1.8314	383	27	267.8692	267.0000	1.8266
384	27	268.5935	267.0000	3.0178	385	27	269.5898	268.0000	3.0100	386	27	270.5834	269.0000	2.9967
387	27	271.5797	270.0000	2.9890	388	27	272.5733	271.0000	2.9758	389	27	273.5696	272.0000	2.9682
390	27	274.2106	273.0000	2.3144	391	27	275.2070	274.0000	2.3086	392	27	276.2007	275.0000	2.2986
393	27	277.1971	276.0000	2.2928	394	27	278.1909	277.0000	2.2829	395	27	279.1873	278.0000	2.2772
396	27	280.1810	279.0000	2.2674	397	27	281.1774	280.0000	2.2617	398	27	281.9977	281.0000	1.9968

TABLE I (*Continued*)
TABLE OF IMPROVED BOUNDS FOR $A(n, d)$

n	d	new	old	ratio	n	d	new	old	ratio	n	d	new	old	ratio
399	27	282.9941	282.0000	1.9918	400	27	283.9880	283.0000	1.9834	401	27	284.9844	284.0000	1.9785
402	27	285.6362	285.0000	1.5542	403	27	286.6327	286.0000	1.5504	404	27	287.6267	287.0000	1.5440
405	27	288.6232	288.0000	1.5402	406	27	289.6172	289.0000	1.5339	407	27	290.6137	290.0000	1.5302
408	27	291.6077	291.0000	1.5239	409	27	292.6042	292.0000	1.5202	410	27	293.1799	293.0000	1.1328
411	27	294.1764	294.0000	1.1301	412	27	295.1706	295.0000	1.1256	413	27	296.1672	296.0000	1.1229
414	27	297.1614	297.0000	1.1184	415	27	298.1580	298.0000	1.1157	416	27	299.1522	299.0000	1.1113
417	27	300.1488	300.0000	1.1087	418	27	301.1431	301.0000	1.1043	419	27	302.1397	302.0000	1.1016
420	27	303.0515	303.0000	1.0364	421	27	304.0481	304.0000	1.0339	314	29	194.2232	194.0000	1.1673
315	29	195.2186	195.0000	1.1636	316	29	196.2099	196.0000	1.1566	317	29	197.2053	197.0000	1.1529
321	29	200.3679	200.2384	1.0939	322	29	201.3595	201.0000	1.2830	323	29	202.3551	202.0000	1.2791
324	29	203.3467	203.0000	1.2717	325	29	204.3424	204.0000	1.2678	326	29	205.3340	205.0000	1.2605
327	29	206.3296	206.0000	1.2567	328	29	207.3213	207.0000	1.2495	329	29	208.3169	207.2384	2.1119
330	29	209.3086	208.0000	2.4771	331	29	210.3043	209.0000	2.4696	332	29	210.9590	210.0000	1.9440
333	29	211.9547	211.0000	1.9382	334	29	212.9466	212.0000	1.9273	335	29	213.9423	213.0000	1.9216
336	29	214.9342	214.0000	1.9108	337	29	215.9299	214.2384	3.2298	338	29	216.5907	215.0000	3.0120
339	29	217.5865	216.0000	3.0033	340	29	218.5786	217.0000	2.9867	341	29	219.5743	218.0000	2.9780
342	29	220.5664	219.0000	2.9617	343	29	221.5622	220.0000	2.9530	344	29	222.3368	221.0000	2.5260
345	29	223.3327	221.2384	4.2701	346	29	224.3248	222.0000	5.0101	347	29	225.3207	223.0000	4.9956
348	29	226.2051	224.0000	4.6111	349	29	227.2010	225.0000	4.5979	350	29	227.9795	226.0000	3.9436
351	29	228.9754	227.0000	3.9324	352	29	229.9677	228.0000	3.9116	353	29	230.9636	228.2384	6.6127
354	29	231.6399	229.0000	6.2327	355	29	232.6358	230.0000	6.2153	356	29	233.6283	231.0000	6.1830
357	29	234.6243	232.0000	6.1657	358	29	235.6168	233.0000	6.1338	359	29	236.6127	234.0000	6.1166
360	29	237.5011	235.0000	5.6612	361	29	238.4971	236.0000	5.6455	362	29	239.1805	237.0000	4.5330
363	29	240.1765	238.0000	4.5207	364	29	241.1692	239.0000	4.4978	365	29	242.1653	240.0000	4.4855
366	29	243.1580	241.0000	4.4629	367	29	244.1540	242.0000	4.4507	368	29	244.8426	243.0000	3.5865
369	29	245.8387	244.0000	3.5769	370	29	246.8316	245.0000	3.5592	371	29	247.8277	246.0000	3.5497
372	29	248.8205	247.0000	3.5321	373	29	249.8167	248.0000	3.5226	374	29	250.5102	249.0000	2.8485
375	29	251.5064	250.0000	2.8410	376	29	252.4994	251.0000	2.8272	377	29	253.4956	252.0000	2.8198
378	29	254.4886	253.0000	2.8061	379	29	255.4848	254.0000	2.7987	380	29	256.2809	255.0000	2.4299
381	29	257.2771	256.0000	2.4236	382	29	258.2702	257.0000	2.4120	383	29	259.2665	258.0000	2.4057
384	29	259.9680	258.0000	3.9123	385	29	260.9643	259.0000	3.9022	386	29	261.9575	260.0000	3.8840
387	29	262.9538	261.0000	3.8740	388	29	263.9471	262.0000	3.8559	389	29	264.9433	263.0000	3.8459
390	29	265.5547	264.0000	2.9377	391	29	266.5510	265.0000	2.9303	392	29	267.5444	266.0000	2.9169
393	29	268.5408	267.0000	2.9095	394	29	269.5342	268.0000	2.8963	395	29	270.5306	269.0000	2.8890
396	29	271.5240	270.0000	2.8759	397	29	272.5203	271.0000	2.8686	398	29	273.3258	272.0000	2.5067
399	29	274.3222	273.0000	2.5004	400	29	275.3157	274.0000	2.4892	401	29	276.3121	275.0000	2.4830
402	29	276.9350	276.0000	1.9119	403	29	277.9315	277.0000	1.9073	404	29	278.9251	278.0000	1.8989
405	29	279.9216	279.0000	1.8942	406	29	280.9153	280.0000	1.8859	407	29	281.9118	281.0000	1.8813
408	29	282.9055	282.0000	1.8731	409	29	283.9019	283.0000	1.8685	410	29	284.4424	284.0000	1.3588
411	29	285.4389	285.0000	1.3556	412	29	286.4327	286.0000	1.3498	413	29	287.4293	287.0000	1.3466
414	29	288.4231	288.0000	1.3409	415	29	289.4197	289.0000	1.3377	416	29	290.4136	290.0000	1.3320
417	29	291.4101	291.0000	1.3288	418	29	292.4040	292.0000	1.3232	419	29	293.4006	293.0000	1.3200
420	29	294.3052	294.0000	1.2356	421	29	295.3018	295.0000	1.2327					

The bound for $\rho_{\inf}(h)$ can be improved as follows. From Corollary 4, Theorem 7, and Lemma 6 follows that

$$\begin{aligned} &A(n, 2h+1) \\ &\quad = A(n+1, 2h+2) \geq \sum_{w \equiv u (\bmod 2h+2)} A(n+1, 2h+2, w) \\ &\quad \geq \frac{1}{c(n+1, 2h+2)} \sum_{w \equiv u (\bmod 2h+2)} \binom{n+1}{w} \\ &\quad = \frac{1}{c(n+1, 2h+2)} \frac{2^{n+1}}{2h+2}(1+o(1)) \\ &\quad = \frac{2^n}{(h+1)n^h}(1+o(1)) \end{aligned}$$

(where $u = \lfloor (n+1)/2 \rfloor$). This bound already appeared in lectures of Gregory Kabatiansky. We have the following theorem.

Theorem 8: We have the following lower bound for $\rho_{\inf}(h)$:

$$\rho_{\inf}(h) \geq \frac{1}{(h+1)!}.$$

V. TABLE OF IMPROVED BOUNDS FOR $A(n, d)$

Table I gives all the improved bounds for $A(n, d)$ where $1 \leq n \leq 512$ and $3 \leq d \leq 29$. The "new" column and the "old" column give the new and old lower bounds for $\log_2 A(n, d)$, respectively. The "ratio" column gives the ratio of the new and old lower bounds for $A(n, d)$.

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